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QUADRATIC OPTIMIZATION UNDER SEMI-FIBONACCI CONSTRAINT (II)

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Abstract It is shown that the Fibonacci sequence is optimal for two quadratic programming problems (maximization and minimization) under semi-Fibonacci constraints. The two conditional (primal) problems have their unconditional (dual) problems. The optimal solution is characterized by the Fibonacci number. Both pairs of primal and dual problems are mutually derived through three methods — dynamic, plus-minus and inequality —.

Keywords: Optimization, quadratic optimization, dualization, semi-Fibonacci constraint

1. Introduction

The *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation,

$$x_{n+2} - x_{n+1} - x_n = 0, \quad x_1 = 1, \ x_0 = 0. \tag{1.1}$$

Table 1: Fibonacci sequence $\{F_n\}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

"The DA VINCI Code" (2006) shows the following ten-digit number:

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 1 \quad 3 \quad 2 \quad 1. \tag{1.2}$$

The code utilizes the Fibonacci sequence as a mysterious code [8], which is the first eight numbers

$$F_1, \quad F_2, \quad F_3, \quad F_4, \quad F_5, \quad F_6, \quad F_7, \quad F_8$$

in the Fibonacci sequence (Table 1).

Recently the Fibonacci Code is constructed in optimal solutions of a primal problem and its dual problem for a general n-variable quadratic optimization [11–17, 19, 20, 25], which are *Fibonacci complementary duality* (FCD) and *Fibonacci shift duality* (FSD). In the case $n = 4$, the Fibonacci Code is right "Da Vinci Code."

In this paper, we consider, as primal problems, two 8-variable quadratic programming problems under semi-Fibonacci constraints. One is minimization. The other is maximization. It turns out that both the problems have "Da Vinci Code" as optimal points. Moreover, this paper discusses how to derive the dual problems from the primal one. We present *three derivation methods* — (1) dynamic, (2) plus-minus and (3) inequality —.

In section 2, we introduce two conditional problems under semi-Fibonacci constraints, which are a minimization problem (P_1) and a maximization problem (P_2). These problems have dual (unconditional) problems (D_1) and (D_2), respectively. We show that optimal solutions of both (primal and dual) problems are characterized by the Fibonacci number. In section 3, we derive (D_1) from (P_1) and vice versa, and propose three dualizations: dynamic method, plus-minus method and inequality method. Section 4 specifies three dualizations between (P_2) and (D_2). Though we discuss 8-variable problems, our results are valid for general $2n$-variable ones.

2. Primal and Dual Problems

We consider an 8-variable conditional problem

$$(\mathrm{P}_1)\quad \begin{array}{lll} \text{minimize} & y_1^2 + y_2^2 + \cdots + y_8^2 & \\ \text{subject to} & \text{(i)}\ \ y_1 + y_2 = y_3 & (\rightsquigarrow\ \nu_1) \\ & \text{(ii)}\ \ y_3 + y_4 = y_5 & (\rightsquigarrow\ \nu_2) \\ & \text{(iii)}\ \ y_5 + y_6 = y_7 & (\rightsquigarrow\ \nu_3) \\ & \text{(iv)}\ \ y_7 + y_8 = c & (\rightsquigarrow\ \nu_4) \\ & \text{(v)}\ \ y \in R^8 & \end{array}$$

where $y = (y_1, y_2, \ldots, y_8)$. The sign "$\rightsquigarrow$" means a corresponding dual variable for an equality constraint. The primal problem (P_1) has a minimum value $m_1 = \dfrac{F_8}{F_9}c^2$ at a minimum point

$$y = (y_1, y_2, \ldots, y_8) = \frac{c}{F_9}(F_1,\ F_2,\ \ldots,\ F_8).$$

In a particular case $c = F_9$, the minimum point y is called the "Da Vinci Code" and the minimum value m_1 is F_8F_9.

The following 4-variable unconditional problem

$$(\mathrm{D}_1)\quad \begin{array}{ll} \text{Maximize} & -\left[2\nu_1^2 + (\nu_2-\nu_1)^2 + \nu_2^2 + (\nu_3-\nu_2)^2 + \nu_3^2 + (\nu_4-\nu_3)^2 + \nu_4^2\right] + 2c\nu_4 \\ \text{subject to} & \text{(i)}\ \ (\nu_1, \nu_2, \nu_3, \nu_4) \in R^4 \end{array}$$

is a dual problem of (P_1). This has a maximum value $M_1 = \dfrac{F_8}{F_9}c^2$ at a maximum point

$$\nu = (\nu_1, \nu_2, \nu_3, \nu_4) = \frac{c}{F_9}(F_2,\ F_4,\ F_6,\ F_8).$$

Next let us consider an 8-variable conditional problem

$$(\mathrm{P}_2)\quad \begin{array}{lll} \text{Maximize} & -(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_5^2 + \mu_6^2 + \mu_7^2 + \mu_8^2) + 2c\mu_8 & \\ \text{subject to} & \text{(i)}\ \ \mu_1 = \mu_2 & (\rightsquigarrow\ x_1) \\ & \text{(ii)}\ \ \mu_2 + \mu_3 = \mu_4 & (\rightsquigarrow\ x_2) \\ & \text{(iii)}\ \ \mu_4 + \mu_5 = \mu_6 & (\rightsquigarrow\ x_3) \\ & \text{(iv)}\ \ \mu_6 + \mu_7 = \mu_8 & (\rightsquigarrow\ x_4) \\ & \text{(v)}\ \ \mu \in R^8 & \end{array}$$

where $\mu = (\mu_1,\, \mu_2,\, \ldots,\, \mu_8)$. The problem ($P_2$) has a maximum value $M_2 = \dfrac{F_8}{F_9}c^2$ at a maximum point

$$\mu = (\mu_1,\, \mu_2,\, \ldots,\, \mu_8) = \frac{c}{F_9}(F_1,\ F_2,\ \ldots,\ F_8).$$

In the case $c = F_9$, the maximum point μ is "Da Vinci Code", too. Moreover, the maximum value M_2 is F_8F_9. An interesting relation between (P_1) and (P_2) is discussed in [18].

Its dual problem is a 4-variable unconditional problem

$$(\mathrm{D}_2)\qquad \begin{array}{ll} \text{minimize} & x_1^2 + (x_2 - x_1)^2 + x_2^2 + (x_3 - x_2)^2 \\ & \qquad + x_3^2 + (x_4 - x_3)^2 + x_4^2 + (c - x_4)^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^4 \end{array}$$

where $x = (x_1,\, x_2,\, x_3,\, x_4)$. The problem ($D_2$) has a minimum value $m_2 = \dfrac{F_8}{F_9}c^2$ at a minimum point

$$x = (x_1,\, x_2,\, x_3,\, x_4) = \frac{c}{F_9}(F_1,\ F_3,\ F_5,\ F_7).$$

3. Derivation (P_1) $\Longleftrightarrow$ (D_1)

Now we show that an 8-variable conditional problem

$$(\mathrm{P}_1)\qquad \begin{array}{lrll} \text{minimize} & \multicolumn{2}{l}{y_1^2 + y_2^2 + \cdots + y_8^2} & \\ \text{subject to} & \text{(i)} & y_1 + y_2 = y_3 & (\rightsquigarrow\ \nu_1) \\ & \text{(ii)} & y_3 + y_4 = y_5 & (\rightsquigarrow\ \nu_2) \\ & \text{(iii)} & y_5 + y_6 = y_7 & (\rightsquigarrow\ \nu_3) \\ & \text{(iv)} & y_7 + y_8 = c & (\rightsquigarrow\ \nu_4) \\ & \text{(v)} & y \in R^8 & \end{array}$$

and a 4-variable unconditional problem

$$(\mathrm{D}_1)\qquad \begin{array}{ll} \text{Maximize} & -\left[\,2\nu_1^2 + (\nu_2 - \nu_1)^2 + \nu_2^2 + (\nu_3 - \nu_2)^2 + \nu_3^2 + (\nu_4 - \nu_3)^2 + \nu_4^2\,\right] + 2c\nu_4 \\ \text{subject to} & \text{(i)}\ \ (\nu_1,\, \nu_2,\, \nu_3,\, \nu_4) \in R^4 \end{array}$$

are dual to each other. We note that the objective function of (P_1) is

$$y_1^2 + y_2^2 + \cdots + y_8^2 = y_1^2 + (y_3 - y_1)^2 + y_3^2 + (y_5 - y_3)^2 + y_5^2 + (y_7 - y_5)^2 + y_7^2 + (c - y_7)^2.$$

Thus the conditional problem (P_1) is equivalent to a 4-variable unconditional problem:

$$(\mathrm{P}_1^*)\qquad \begin{array}{ll} \text{minimize} & y_1^2 + (y_3 - y_1)^2 + y_3^2 + (y_5 - y_3)^2 \\ & \qquad + y_5^2 + (y_7 - y_5)^2 + y_7^2 + (c - y_7)^2 \\ \text{subject to} & \text{(i)}\ \ (y_1,\, y_3,\, y_5,\, y_7) \in R^4. \end{array}$$

A two-way derivation is presented by three methods — (1) dynamic, (2) plus-minus and (3) inequality —.

3.1. Dynamic method

This method is basically an expansion of the Lagrangian method for constraint optimization. The dynamic method is applicable to unconstrained optimization [13]. The method for constraint is reduced to the Lagrangian one.

Let $\nu_1, \ldots, \nu_4$ be a dual variables corresponding to constraints (i), …, (iv) in (P_1), respectively. Let $y = (y_1,\, y_2,\, \ldots,\, y_8) \in R^8$ be any feasible solution to (P_1). Then we have

for any $\nu = (\nu_1,\, \nu_2,\, \nu_3,\, \nu_4) \in R^4$

$$\begin{aligned}
& y_1^2 + y_2^2 + \cdots + y_8^2 \\
= \;& y_1^2 + y_2^2 + \cdots + y_8^2 + 2\nu_1(y_3 - y_1 - y_2) + 2\nu_2(y_5 - y_3 - y_4) + 2\nu_3(y_7 - y_5 - y_6) \\
& + 2\nu_4(c - y_7 - y_8) \\
= \;& y_1^2 - 2\nu_1 y_1 + y_2^2 - 2\nu_1 y_2 + y_3^2 - 2(\nu_2 - \nu_1)y_3 + y_4^2 - 2\nu_2 y_4 \\
& + y_5^2 - 2(\nu_3 - \nu_2)y_5 + y_6^2 - 2\nu_3 y_6 + y_7^2 - 2(\nu_4 - \nu_3)y_7 + y_8^2 - 2\nu_4 y_8 + 2c\nu_4 \\
= \;& (y_1 - \nu_1)^2 - \nu_1^2 + (y_2 - \nu_1)^2 - \nu_1^2 + \{y_3 - (\nu_2 - \nu_1)\}^2 - (\nu_2 - \nu_1)^2 \\
& + (y_4 - \nu_2)^2 - \nu_2^2 + \{y_5 - (\nu_3 - \nu_2)\}^2 - (\nu_3 - \nu_2)^2 + (y_6 - \nu_3)^2 - \nu_3^2 \\
& + \{y_7 - (\nu_4 - \nu_3)\}^2 - (\nu_4 - \nu_3)^2 + (y_8 - \nu_4)^2 - \nu_4^2 + 2c\nu_4 \\
\geq \;& -\left[2\nu_1^2 + (\nu_2 - \nu_1)^2 + \nu_2^2 + (\nu_3 - \nu_2)^2 + \nu_3^2 + (\nu_4 - \nu_3)^2 + \nu_4^2 \right] + 2c\nu_4.
\end{aligned}$$

The sign of equality holds iff

$$\begin{aligned}
y_1 &= \nu_1, & y_2 &= \nu_1 \\
y_3 &= \nu_2 - \nu_1, & y_4 &= \nu_2 \\
y_5 &= \nu_3 - \nu_2, & y_6 &= \nu_3 \\
y_7 &= \nu_4 - \nu_3, & y_8 &= \nu_4
\end{aligned} \tag{3.1}$$

hold. Hence we have for y satisfying (i) $\sim$ (iv) and ν

$$y_1^2 + y_2^2 + \cdots + y_8^2 \geq -\left[2\nu_1^2 + (\nu_2 - \nu_1)^2 + \nu_2^2 + (\nu_3 - \nu_2)^2 + \nu_3^2 + (\nu_4 - \nu_3)^2 + \nu_4^2 \right] + 2c\nu_4. \tag{3.2}$$

The sign of equality holds iff (3.1) holds together with (i) $\sim$ (iv). This system — 12 linear equations in 12 variables — is equivalent to

$$\begin{aligned}
y_2 &= \nu_1, & y_4 &= \nu_2 \\
y_6 &= \nu_3, & y_8 &= \nu_4 \\
y_1 - y_2 &= 0, & y_1 + y_2 - y_3 &= 0 \\
y_2 + y_3 - y_4 &= 0, & y_3 + y_4 - y_5 &= 0 \\
y_4 + y_5 - y_6 &= 0, & y_5 + y_6 - y_7 &= 0 \\
y_6 + y_7 - y_8 &= 0, & y_7 + y_8 - c &= 0.
\end{aligned} \tag{3.3}$$

The system has a unique solution

$$\begin{aligned}
(y_1,\ y_2,\ \ldots,\ y_8) &= \frac{c}{F_9}(F_1,\ F_2,\ \ldots,\ F_8)\,; \\
(\nu_1,\ \nu_2,\ \nu_3,\ \nu_4) &= \frac{c}{F_9}(F_2,\ F_4,\ F_6,\ F_8).
\end{aligned} \tag{3.4}$$

Thus (D_1) is derived.

Conversely (P_1) is derived by tracing the argument back. Let $\nu \in R^4$ be any feasible solution to (D_1). For any $y \in R^8$, we have the inequality

$$\begin{aligned}
&-\left[\,2\nu_1^2+(\nu_2-\nu_1)^2+\nu_2^2+(\nu_3-\nu_2)^2+\nu_3^2+(\nu_4-\nu_3)^2+\nu_4^2\,\right]+2c\nu_4\\
\le\;& (y_1-\nu_1)^2-\nu_1^2+(y_2-\nu_1)^2-\nu_1^2+\{y_3-(\nu_2-\nu_1)\}^2-(\nu_2-\nu_1)^2\\
&+(y_4-\nu_2)^2-\nu_2^2+\{y_5-(\nu_3-\nu_2)\}^2-(\nu_3-\nu_2)^2+(y_6-\nu_3)^2-\nu_3^2\\
&+\{y_7-(\nu_4-\nu_3)\}^2-(\nu_4-\nu_3)^2+(y_8-\nu_4)^2-\nu_4^2+2c\nu_4. \qquad (3.5)
\end{aligned}$$

The sign of equality holds iff (3.1) holds. Moreover, the right hand side in (3.5) becomes

$$\begin{aligned}
&y_1^2-2\nu_1y_1+y_2^2-2\nu_1y_2+y_3^2-2(\nu_2-\nu_1)y_3+y_4^2-2\nu_2y_4\\
&\quad+y_5^2-2(\nu_3-\nu_2)y_5+y_6^2-2\nu_3y_6+y_7^2-2(\nu_4-\nu_3)y_7+y_8^2-2\nu_4y_8+2c\nu_4\\
=\;& y_1^2+y_2^2+\cdots+y_8^2\\
&\quad+2\nu_1(y_3-y_1-y_2)+2\nu_2(y_5-y_3-y_4)+2\nu_3(y_7-y_5-y_6)+2\nu_4(c-y_7-y_8).
\end{aligned}$$

In particular, under the condition (i) $\sim$ (iv) we get the equality

$$\begin{aligned}
&y_1^2+y_2^2+\cdots+y_8^2\\
&\quad+2\nu_1(y_3-y_1-y_2)+2\nu_2(y_5-y_3-y_4)+2\nu_3(y_7-y_5-y_6)+2\nu_4(c-y_7-y_8)\\
=\;& y_1^2+y_2^2+\cdots+y_8^2.
\end{aligned}$$

Hence the inequality (3.2) holds for any ν and y satisfying (i) $\sim$ (iv). The sign of equality holds iff (3.1) holds. The condition (3.1) and (i) $\sim$ (iv) constitute a system (3.3). The system has a unique solution (3.4). Thus (P_1) is derived.

3.2. Plus-minus method

This method is based upon Fenchel duality[9]. The plus-minus method is viewed as an application of maximum transform, quasi-linearization, or conjugate function [2–7, 10, 23, 24].

Let $y \in R^8$ be any feasible solution to (P_1). For any $\nu \in R^4$, we have the inequality

$$\begin{aligned}
&y_1^2+y_2^2+\cdots+y_8^2\\
=\;& y_1^2+y_2^2+\cdots+y_8^2\\
&-2\nu_1y_1-2\nu_1y_2-2(\nu_2-\nu_1)y_3-2\nu_2y_4-2(\nu_3-\nu_2)y_5-2\nu_3y_6-2(\nu_4-\nu_3)y_7-2\nu_4y_8\\
&+2\nu_1y_1+2\nu_1y_2+2(\nu_2-\nu_1)y_3+2\nu_2y_4+2(\nu_3-\nu_2)y_5+2\nu_3y_6+2(\nu_4-\nu_3)y_7+2\nu_4y_8\\
=\;& (y_1-\nu_1)^2-\nu_1^2+(y_2-\nu_1)^2-\nu_1^2+\{y_3-(\nu_2-\nu_1)\}^2-(\nu_2-\nu_1)^2\\
&+(y_4-\nu_2)^2-\nu_2^2+\{y_5-(\nu_3-\nu_2)\}^2-(\nu_3-\nu_2)^2+(y_6-\nu_3)^2-\nu_3^2\\
&+\{y_7-(\nu_4-\nu_3)\}^2-(\nu_4-\nu_3)^2+(y_8-\nu_4)^2-\nu_4^2+2c\nu_4\\
\ge\;& -\left[\,2\nu_1^2+(\nu_2-\nu_1)^2+\nu_2^2+(\nu_3-\nu_2)^2+\nu_3^2+(\nu_4-\nu_3)^2+\nu_4^2\,\right]+2c\nu_4.
\end{aligned}$$

The sign of equality holds iff (3.1) holds. The condition (3.1) and (i) $\sim$ (iv) constitute a system (3.3). The system has a unique solution (3.4). Thus (D_1) is derived.

Conversely (P_1) is derived. In fact, when the unconditional problem

$$(\mathrm{D}_1)\quad \begin{array}{ll} \text{Maximize} & -\left[\,2\nu_1^2+(\nu_2-\nu_1)^2+\nu_2^2+(\nu_3-\nu_2)^2+\nu_3^2+(\nu_4-\nu_3)^2+\nu_4^2\,\right]+2c\nu_4\\ \text{subject to} & \text{(i)}\;\; (\nu_1,\,\nu_2,\,\nu_3,\,\nu_4)\in R^4 \end{array}$$

is given, the conditional problem

$$
(\mathrm{P}_1)\qquad
\begin{array}{lll}
\text{minimize} & y_1^2 + y_2^2 + \cdots + y_8^2 & \\
\text{subject to} & \text{(i)}\ \ y_1 + y_2 = y_3 & (\rightsquigarrow\ \nu_1) \\
 & \text{(ii)}\ \ y_3 + y_4 = y_5 & (\rightsquigarrow\ \nu_2) \\
 & \text{(iii)}\ \ y_5 + y_6 = y_7 & (\rightsquigarrow\ \nu_3) \\
 & \text{(iv)}\ \ y_7 + y_8 = c & (\rightsquigarrow\ \nu_4) \\
 & \text{(v)}\ \ y \in R^8 &
\end{array}
$$

is derived. To show this, we establish an inequality (3.2) for any ν and y satisfying (i) $\sim$ (iv), and show that the sign of equality holds iff (3.1) holds.

Now let $\nu \in R^4$ be any feasible solution to (D_1). Then we have for any $y \in R^8$

$$
\begin{aligned}
& -\left[\,2\nu_1^2 + (\nu_2-\nu_1)^2 + \nu_2^2 + (\nu_3-\nu_2)^2 + \nu_3^2 + (\nu_4-\nu_3)^2 + \nu_4^2\,\right] + 2c\nu_4 \\
= \ & -\left[\,2\nu_1^2 + (\nu_2-\nu_1)^2 + \nu_2^2 + (\nu_3-\nu_2)^2 + \nu_3^2 + (\nu_4-\nu_3)^2 + \nu_4^2\,\right] + 2c\nu_4 \\
& + 2y_1\nu_1 + 2y_2\nu_1 + 2y_3(\nu_2-\nu_1) + 2y_4\nu_2 + 2y_5(\nu_3-\nu_2) + 2y_6\nu_3 + 2y_7(\nu_4-\nu_3) + 2y_8\nu_4 \\
& - 2y_1\nu_1 - 2y_2\nu_1 - 2y_3(\nu_2-\nu_1) - 2y_4\nu_2 - 2y_5(\nu_3-\nu_2) - 2y_6\nu_3 - 2y_7(\nu_4-\nu_3) - 2y_8\nu_4 \\
= \ & -(\nu_1-y_1)^2 + y_1^2 - (\nu_1-y_2)^2 + y_2^2 - (\nu_2-\nu_1-y_3)^2 + y_3^2 \\
& - (\nu_2-y_4)^2 + y_4^2 - (\nu_3-\nu_2-y_5)^2 + y_5^2 - (\nu_3-y_6)^2 + y_6^2 \\
& - (\nu_4-\nu_3-y_7)^2 + y_7^2 - (\nu_4-y_8)^2 + y_8^2 \\
& - 2(y_1+y_2-y_3)\nu_1 - 2(y_3+y_4-y_5)\nu_2 - 2(y_5+y_6-y_7)\nu_3 - 2(y_7+y_8-c)\nu_4.
\end{aligned}
$$

In particular, under the condition (i) $\sim$ (iv) we obtain the inequality

$$
\begin{aligned}
& -(\nu_1-y_1)^2 + y_1^2 - (\nu_1-y_2)^2 + y_2^2 - (\nu_2-\nu_1-y_3)^2 + y_3^2 - (\nu_2-y_4)^2 + y_4^2 \\
& \quad - (\nu_3-\nu_2-y_5)^2 + y_5^2 - (\nu_3-y_6)^2 + y_6^2 - (\nu_4-\nu_3-y_7)^2 + y_7^2 - (\nu_4-y_8)^2 + y_8^2 \\
\le \ & y_1^2 + y_2^2 + \cdots + y_8^2.
\end{aligned}
$$

The sign of equality holds iff (3.1) holds. Hence we have the inequality (3.2) for any ν and y satisfying (i) $\sim$ (iv). The condition (3.1) and (i) $\sim$ (iv) constitute a system (3.3). The system has a unique solution (3.4). Thus (P_1) is derived.

3.3. Inequality method

By applying an inequality, we show that (P_1) and (D_1) are dual to each other. Our *inequality approach* [13] is based upon the elementary inequality with equality condition:

$$
2xy \le x^2 + y^2 \ \text{ on } R^2\ ; \quad x = y. \tag{3.6}
$$

This is equivalent to Arithmetic-mean/Geometric-mean inequality.

Now let us show a two-way derivation $(\mathrm{P}_1) \iff (\mathrm{D}_1)$. Let $y = (y_1, y_2, \ldots, y_8) \in R^8$ be feasible for (P_1) and $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in R^4$ be feasible for (D_1). That is, we assume that y satisfies the constraint (i) $\sim$ (iv).

By applying the inequality (3.6), we have eight inequalities :

$$\begin{aligned}
2y_1\nu_1 &\le y_1^2+\nu_1^2\ ; \quad y_1=\nu_1\\
2y_2\nu_1 &\le y_2^2+\nu_1^2\ ; \quad y_2=\nu_1\\
2y_3(\nu_2-\nu_1) &\le y_3^2+(\nu_2-\nu_1)^2\ ; \quad y_3=\nu_2-\nu_1\\
2y_4\nu_2 &\le y_4^2+\nu_2^2\ ; \quad y_4=\nu_2\\
2y_5(\nu_3-\nu_2) &\le y_5^2+(\nu_3-\nu_2)^2\ ; \quad y_5=\nu_3-\nu_2\\
2y_6\nu_3 &\le y_6^2+\nu_3^2\ ; \quad y_6=\nu_3\\
2y_7(\nu_4-\nu_3) &\le y_7^2+(\nu_4-\nu_3)^2\ ; \quad y_7=\nu_4-\nu_3\\
2y_8\nu_4 &\le y_8^2+\nu_4^2\ ; \quad y_8=\nu_4.
\end{aligned}$$

Summing up both sides, we get

$$\begin{aligned}
&2\big[\,y_1\nu_1+y_2\nu_1+y_3(\nu_2-\nu_1)+y_4\nu_2\\
&\qquad\qquad +y_5(\nu_3-\nu_2)+y_6\nu_3+y_7(\nu_4-\nu_3)+y_8\nu_4\,\big]\\
\le\ &\sum_{k=1}^{8} y_k^2+\big[\,2\nu_1^2+(\nu_2-\nu_1)^2+\nu_2^2+(\nu_3-\nu_2)^2+\nu_3^2+(\nu_4-\nu_3)^2+\nu_4^2\,\big]
\end{aligned}$$

with an equality condition :

$$\text{(e)}\quad \begin{array}{l} y_1=\nu_1,\quad y_2=\nu_1,\quad y_3=\nu_2-\nu_1,\quad y_4=\nu_2,\\ y_5=\nu_3-\nu_2,\quad y_6=\nu_3,\quad y_7=\nu_4-\nu_3,\quad y_8=\nu_4. \end{array}$$

The condition (i) $\sim$ (iv) implies that

$$\begin{aligned}
&y_1\nu_1+y_2\nu_1+y_3(\nu_2-\nu_1)+y_4\nu_2+y_5(\nu_3-\nu_2)+y_6\nu_3+y_7(\nu_4-\nu_3)+y_8\nu_4\\
=\ &(y_1+y_2-y_3)\nu_1+(y_3+y_4-y_5)\nu_2+(y_5+y_6-y_7)\nu_3+(y_7+y_8-c)\nu_4+c\nu_4\\
=\ &c\nu_4.
\end{aligned}$$

Thus we get an inequality

$$2c\nu_4 \le \sum_{1}^{8} y_k^2+\big[\,2\nu_1^2+(\nu_2-\nu_1)^2+\nu_2^2+(\nu_3-\nu_2)^2+\nu_3^2+(\nu_4-\nu_3)^2+\nu_4^2\,\big].$$

The sign of equality holds iff (e) holds. Hence it holds that

$$-\big[\,2\nu_1^2+(\nu_2-\nu_1)^2+\nu_2^2+(\nu_3-\nu_2)^2+\nu_3^2+(\nu_4-\nu_3)^2+\nu_4^2\,\big]+2c\nu_4\le y_1^2+\cdots+y_8^2$$

for any feasible ν and y. The sign of equality holds iff (i) $\sim$ (iv), (e) hold :

$$\begin{aligned}
y_1 &= \nu_1, & y_2 &= \nu_1\\
y_3 &= \nu_2-\nu_1, & y_4 &= \nu_2\\
y_5 &= \nu_3-\nu_2, & y_6 &= \nu_3\\
y_7 &= \nu_4-\nu_3, & y_8 &= \nu_4\\
y_1+y_2-y_3 &= 0, & y_3+y_4-y_5 &= 0\\
y_5+y_6-y_7 &= 0, & y_7+y_8-c &= 0.
\end{aligned}$$

This system is equivalent to (3.3). The system has a unique solution (3.4). Thus both problems are dual to each other.

4. Derivation $(\mathrm{P}_2) \Longleftrightarrow (\mathrm{D}_2)$

We note that the objective function of (P_2) is

$$\begin{aligned}
&-(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8 \\
=&\ -\left[\,2\mu_2^2+(\mu_4-\mu_2)^2+\mu_4^2+(\mu_6-\mu_4)^2+\mu_6^2+(\mu_8-\mu_6)^2+\mu_8^2\,\right]+2c\mu_8.
\end{aligned}$$

Thus the 8-variable conditional problem (P_2) is equivalent to a 4-variable unconditional problem:

$$(\mathrm{P}_2^*)\qquad \begin{aligned}
&\text{Maximize} && -\left[\,2\mu_2^2+(\mu_4-\mu_2)^2+\mu_4^2+(\mu_6-\mu_4)^2\right. \\
& && \qquad\qquad \left.+\,\mu_6^2+(\mu_8-\mu_6)^2+\mu_8^2\,\right]+2c\mu_8 \\
&\text{subject to} && \text{(i)}\ \ (\mu_2,\,\mu_4,\,\mu_6,\,\mu_8)\in R^4.
\end{aligned}$$

4.1. Dynamic method

Next we show that a dual problem of the (primal) maximization problem (P_2) is a minimization problem (D_2). Let $\mu=(\mu_1,\,\mu_2,\,\ldots,\,\mu_8)\in R^8$ be any feasible solution to (P_2). For any $x=(x_1,\,x_2,\,x_3,\,x_4)\in R^4$, we get the inequality

$$\begin{aligned}
&-(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8 \\
=&\ -(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8 \\
&+2x_1(\mu_1-\mu_2)+2x_2(\mu_2+\mu_3-\mu_4)+2x_3(\mu_4+\mu_5-\mu_6)+2x_4(\mu_6+\mu_7-\mu_8) \\
=&\ -\mu_1^2+2x_1\mu_1-\mu_2^2+2(x_2-x_1)\mu_2-\mu_3^2+2x_2\mu_3-\mu_4^2+2(x_3-x_2)\mu_4 \\
&-\mu_5^2+2x_3\mu_5-\mu_6^2+2(x_4-x_3)\mu_6-\mu_7^2+2x_4\mu_7-\mu_8^2+2(c-x_4)\mu_8 \\
=&\ -(\mu_1-x_1)^2+x_1^2-\{\mu_2-(x_2-x_1)\}^2+(x_2-x_1)^2 \\
&-(\mu_3-x_2)^2+x_2^2-\{\mu_4-(x_3-x_2)\}^2+(x_3-x_2)^2 \\
&-(\mu_5-x_3)^2+x_3^2-\{\mu_6-(x_4-x_3)\}^2+(x_4-x_3)^2 \\
&-(\mu_7-x_4)^2+x_4^2-\{\mu_8-(c-x_4)\}^2+(c-x_4)^2 \\
\le&\ x_1^2+(x_2-x_1)^2+x_2^2+(x_3-x_2)^2+x_3^2+(x_4-x_3)^2+x_4^2+(c-x_4)^2.
\end{aligned}$$

The sign of equality holds iff

$$\begin{aligned}
\mu_1 &= x_1, & \mu_2 &= x_2-x_1 \\
\mu_3 &= x_2, & \mu_4 &= x_3-x_2 \\
\mu_5 &= x_3, & \mu_6 &= x_4-x_3 \\
\mu_7 &= x_4, & \mu_8 &= c-x_4
\end{aligned} \tag{4.1}$$

hold.

Hence we have for μ satisfying (i) $\sim$ (iv) and x

$$-(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8 \le x_1^2+(x_2-x_1)^2+x_2^2+(x_3-x_2)^2+x_3^2+(x_4-x_3)^2+x_4^2+(c-x_4)^2. \tag{4.2}$$

The sign of equality holds iff (4.1) holds together with (i) $\sim$ (iv). This system — 12 linear equations in 12 variables — is equivalent to

$$\begin{aligned} \mu_1 &= x_1, & \mu_3 &= x_2 \\ \mu_5 &= x_3, & \mu_7 &= x_4 \\ \mu_1 - \mu_2 &= 0, & \mu_1 + \mu_2 - \mu_3 &= 0 \\ \mu_2 + \mu_3 - \mu_4 &= 0, & \mu_3 + \mu_4 - \mu_5 &= 0 \\ \mu_4 + \mu_5 - \mu_6 &= 0, & \mu_5 + \mu_6 - \mu_7 &= 0 \\ \mu_6 + \mu_7 - \mu_8 &= 0, & \mu_7 + \mu_8 - c &= 0. \end{aligned} \tag{4.3}$$

The system has a unique solution

$$\begin{aligned} (\mu_1,\ \mu_2,\ \ldots,\ \mu_8) &= \frac{c}{F_9}(F_1,\ F_2,\ \ldots,\ F_8)\,; \\ (x_1,\ x_2,\ x_3,\ x_4) &= \frac{c}{F_9}(F_1,\ F_3,\ F_5,\ F_7). \end{aligned} \tag{4.4}$$

Thus (D_2) is derived.

Conversely (P_2) is derived by tracing the argument back. Let $x \in R^4$ be any feasible solution to (D_2). For any $\mu \in R^8$, we get the inequality

$$\begin{aligned} & x_1^2 + (x_2 - x_1)^2 + x_2^2 + (x_3 - x_2)^2 + x_3^2 + (x_4 - x_3)^2 + x_4^2 + (c - x_4)^2 \\ \geq\ & -(\mu_1 - x_1)^2 + x_1^2 - \{\mu_2 - (x_2 - x_1)\}^2 + (x_2 - x_1)^2 \\ & - (\mu_3 - x_2)^2 + x_2^2 - \{\mu_4 - (x_3 - x_2)\}^2 + (x_3 - x_2)^2 \\ & - (\mu_5 - x_3)^2 + x_3^2 - \{\mu_6 - (x_4 - x_3)\}^2 + (x_4 - x_3)^2 \\ & - (\mu_7 - x_4)^2 + x_4^2 - \{\mu_8 - (c - x_4)\}^2 + (c - x_4)^2. \end{aligned} \tag{4.5}$$

The sign of equality holds iff (4.1) holds. Moreover, the right hand side in (4.5) becomes

$$\begin{aligned} & -(\mu_1 - x_1)^2 + x_1^2 - \{\mu_2 - (x_2 - x_1)\}^2 + (x_2 - x_1)^2 - \cdots - \{\mu_8 - (c - x_4)\}^2 + (c - x_4)^2 \\ =\ & -(\mu_1^2 + \mu_2^2 + \cdots + \mu_8^2) + 2c\mu_8 \\ & + 2x_1(\mu_1 - \mu_2) + 2x_2(\mu_2 + \mu_3 - \mu_4) + 2x_3(\mu_4 + \mu_5 - \mu_6) + 2x_4(\mu_6 + \mu_7 - \mu_8). \end{aligned}$$

Under the condition (i) $\sim$ (iv) we obtain

$$\begin{aligned} & -(\mu_1 - x_1)^2 + x_1^2 - \{\mu_2 - (x_2 - x_1)\}^2 + (x_2 - x_1)^2 - \cdots - \{\mu_8 - (c - x_4)\}^2 + (c - x_4)^2 \\ =\ & -(\mu_1^2 + \mu_2^2 + \cdots + \mu_8^2) + 2c\mu_8. \end{aligned}$$

Hence the inequality (4.2) holds for any x and μ satisfying (i) $\sim$ (iv). The sign of equality holds iff (4.1) holds. The condition (4.1) and (i) $\sim$ (iv) constitute a system (4.3). The system has a unique solution (4.4). Thus (P_2) is derived.

4.2. Plus-minus method

Let $\mu \in R^8$ be any feasible solution to (P$_2$). Then we have for any $x \in R^4$

$$\begin{aligned}
&-(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8\\
=\;& -(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8\\
&+2x_1\mu_1+2x_2\mu_3+2x_3\mu_5+2x_4\mu_7-2x_1\mu_1-2x_2\mu_3-2x_3\mu_5-2x_4\mu_7\\
=\;& -(\mu_1-x_1)^2+x_1^2-\{\mu_2-(x_2-x_1)\}^2+(x_2-x_1)^2\\
&-(\mu_3-x_2)^2+x_2^2-\{\mu_4-(x_3-x_2)\}^2+(x_3-x_2)^2\\
&-(\mu_5-x_3)^2+x_3^2-\{\mu_6-(x_4-x_3)\}^2+(x_4-x_3)^2\\
&-(\mu_7-x_4)^2+x_4^2-\{\mu_8-(c-x_4)\}^2+(c-x_4)^2\\
\le\;& x_1^2+(x_2-x_1)^2+x_2^2+(x_3-x_2)^2+x_3^2+(x_4-x_3)^2+x_4^2+(c-x_4)^2.
\end{aligned}$$

The sign of equality holds iff the condition (4.1) holds. The condition (4.1) and (i) $\sim$ (iv) constitute a system (4.3). The system has a unique solution (4.4). Thus (D$_2$) is derived.

Conversely (P$_2$) is derived. When the unconditional problem (D$_2$) is given, the conditional problem (P$_2$) is derived. To show this, we establish an inequality (4.2) for any x and μ satisfying (i) $\sim$ (iv), and show that the sign of equality holds iff (4.1) holds.

Let $x \in R^4$ be any feasible solution to the minimization problem (D$_2$). Then we have for any $\mu \in R^8$

$$\begin{aligned}
&x_1^2+(x_2-x_1)^2+x_2^2+(x_3-x_2)^2+x_3^2+(x_4-x_3)^2+x_4^2+(c-x_4)^2\\
=\;& x_1^2+(x_2-x_1)^2+x_2^2+(x_3-x_2)^2+x_3^2+(x_4-x_3)^2+x_4^2+(c-x_4)^2\\
&-2\mu_1x_1-2\mu_2(x_2-x_1)-2\mu_3x_2-2\mu_4(x_3-x_2)\\
&-2\mu_5x_3-2\mu_6(x_4-x_3)-2\mu_7x_4-2\mu_8(c-x_4)\\
&+2\mu_1x_1+2\mu_2(x_2-x_1)+2\mu_3x_2-2\mu_4(x_3-x_2)\\
&+2\mu_5x_3+2\mu_6(x_4-x_3)+2\mu_7x_4+2\mu_8(c-x_4)\\
=\;& (x_1-\mu_1)^2-\mu_1^2+(x_2-x_1-\mu_2)^2-\mu_2^2\\
&+(x_2-\mu_3)^2-\mu_3^2+(x_3-x_2-\mu_4)^2-\mu_4^2\\
&+(x_3-\mu_5)^2-\mu_5^2+(x_4-x_3-\mu_6)^2-\mu_6^2\\
&+(x_4-\mu_7)^2-\mu_7^2+(c-x_4-\mu_8)^2-\mu_8^2\\
&+2(\mu_1-\mu_2)x_1+2(\mu_2+\mu_3-\mu_4)x_2+2(\mu_4+\mu_5-\mu_6)x_3+2(\mu_6+\mu_7-\mu_8)x_4+2c\mu_8.
\end{aligned}$$

Under the condition (i) $\sim$ (iv) we obtain an inequality

$$\begin{aligned}
&x_1^2+(x_2-x_1)^2+x_2^2+(x_3-x_2)^2+x_3^2+(x_4-x_3)^2+x_4^2+(c-x_4)^2\\
\ge\;& -(\mu_1^2+\mu_2^2+\cdots+\mu_8^2)+2c\mu_8
\end{aligned}$$

for any x and μ satisfying (i) $\sim$ (iv). The sign of equality holds iff (4.1) holds. The condition (4.1) and (i) $\sim$ (iv) constitute a system (4.3). The system has a unique solution (4.4). Thus (P$_2$) is derived.

4.3. Inequality method

By applying the Arithmetic-mean/Geometric-mean inequality (3.6), we show that (P$_2$) and (D$_2$) are dual to each other.

Now let us show a two-way derivation $(\mathrm{P}_2) \iff (\mathrm{D}_2)$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_8) \in R^8$ be feasible for (P_2) and $x = (x_1, x_2, x_3, x_4) \in R^4$ be feasible for (D_2). That is, we assume that μ satisfies the constraint (i) $\sim$ (iv).

By applying the inequality 8 times and summing over $k = 1, 2, \ldots, 8$, we have an inequality :

$$\begin{aligned} & 2\big[\,\mu_1 x_1 + \mu_2(x_2 - x_1) + \mu_3 x_2 + \mu_4(x_3 - x_2) \\ & \qquad\qquad + \mu_5 x_3 + \mu_6(x_4 - x_3) + \mu_7 x_4 + \mu_8(c - x_4)\,\big] \\ \leq\ & \sum_{k=1}^{8} \mu_k^2 + \big[\,x_1^2 + (x_2 - x_1)^2 + x_2^2 + (x_3 - x_2)^2 + x_3^2 + (x_4 - x_3)^2 + x_4^2 + (c - x_4)^2\,\big] \end{aligned}$$

with an equality condition :

$$(\mathrm{e}') \quad \begin{array}{llll} \mu_1 = x_1, & \mu_2 = x_2 - x_1, & \mu_3 = x_2, & \mu_4 = x_3 - x_2, \\ \mu_5 = x_3, & \mu_5 = x_4 - x_3, & \mu_6 = x_4, & \mu_8 = c - x_4. \end{array}$$

The condition (i) $\sim$ (iv) implies that

$$\begin{aligned} & \mu_1 x_1 + \mu_2(x_2 - x_1) + \mu_3 x_2 + \mu_4(x_3 - x_2) + \mu_5 x_3 + \mu_6(x_4 - x_3) + \mu_7 x_4 + \mu_8(c - x_4) \\ =\ & (\mu_1 - \mu_2)x_1 + (\mu_2 + \mu_3 - \mu_4)x_2 + (\mu_4 + \mu_5 - \mu_6)x_3 + (\mu_6 + \mu_7 - \mu_8)x_4 + c\mu_8 \\ =\ & c\mu_8. \end{aligned}$$

Thus we get an inequality

$$2c\mu_8 \ \leq\ \sum_{1}^{8} \mu_k^2 + \big[\,x_1^2 + (x_2 - x_1)^2 + x_2^2 + (x_3 - x_2)^2 + x_3^2 + (x_4 - x_3)^2 + x_4^2 + (c - x_4)^2\,\big].$$

The sign of equality holds iff (e') holds. Hence it holds that

$$-(\mu_1^2 + \mu_2^2 + \cdots + \mu_8^2) + 2c\mu_8 \leq x_1^2 + (x_2 - x_1)^2 + x_2^2 + (x_3 - x_2)^2 + x_3^2 + (x_4 - x_3)^2 + x_4^2 + (c - x_4)^2$$

for any feasible μ and x. The sign of equality holds iff (i) $\sim$ (iv), (e') hold :

$$\begin{aligned} \mu_1 &= x_1, & \mu_2 &= x_2 - x_1 \\ \mu_3 &= x_2, & \mu_4 &= x_3 - x_2 \\ \mu_5 &= x_3, & \mu_6 &= x_4 - x_3 \\ \mu_7 &= x_4, & \mu_8 &= c - x_4 \\ \mu_1 - \mu_2 &= 0, & \mu_2 + \mu_3 - \mu_4 &= 0 \\ \mu_4 + \mu_5 - \mu_6 &= 0, & \mu_6 + \mu_7 - \mu_8 &= 0. \end{aligned}$$

This system is equivalent to (4.3). The system has a unique solution (4.4). Thus both are dual to each other.

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