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HERGLOTZ-BOCHNER REPRESENTATION THEOREM VIA THEORY OF DISTRIBUTIONS

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Abstract Any positive semi-definite function defined on $\mathbb{Z}$ (resp. $\mathbb{R}$) can be represented as the Fourier transform of a positive Radon measure on $\mathbb{T}$ (resp. $\mathbb{R}$). We give a proof of this celebrated result due to Herglotz and Bochner from the viewpoint of Schwartz's theory of distributions.

Keywords: Applied probability, periodic distribution, positive semi-definite, tempered distribution, Fourier transform of measures

1. Introduction

Any complex-valued function defined on $\mathbb{Z}$ (resp. $\mathbb{R}$) can be represented as the Fourier transform of a positive Radon measure on $\mathbb{T}$ (resp. $\mathbb{R}$) if and only if it is positive semi-definite.[1] This classical theorem due to Herglotz[5] and Bochner[1] has been providing various fields of analysis with a powerful apparatus. For instance, a periodic or almost periodic behavior of a weakly stationary stochastic process can be characterized in terms of the discreteness of its spectral measure, the Fourier transform of which represents the covariance (cf. section 6).

On the other hand, much interests have been attracted to clarify the mathematical structure of the Herglotz-Bochner theorem itself. Several different approaches to the proof of the theorem were explored in the course of the investigations :

Approach 1 – having recourse to the Féjer-Poisson summation method in classical Fourier analysis.

Approach 2 – building a bridge from Stone's representation theorem of one-parameter semi-group of operators.

Approach 3 – making use of abstract theories of normed algebra.

In any case, there seems no easy and quick way leading to the Herglotz-Bochner theorem.

However we should remind of the fourth approach based upon the theory of distributions due to Schwartz[11]. Although we can not say that this approach is very elementary, we may assume that the distribution theory is a common knowledge among most mathematicians. Comparing with other approaches, the distribution approach seems to supply a relatively simple and transparent proof, provided that Schwartz's theory is a common knowledge.

The purpose of this paper is to present a systematic exposition of this approach to the Herglotz-Bochner theorem. I do not intend to claim any novelty taking account of the outline

[1]Throughout this paper, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers and complex numbers, respectively. $\mathbb{T}$ is the torus $\mathbb{R}/2\pi\mathbb{Z}$.

of reasonings described by Katznelson[6] , pp.48-49 and Lax[7], pp.141-147. Nevertheless, there seems no coherent exposition of the approach as far as I know. I expect this brief note to fill the gap and benefit the readers to some extent.

2. Periodic Distributions

A locally integrable function $f \in \mathfrak{L}^1_{loc.}(\mathbb{R}, \mathbb{C})$ defines a distribution T_f by the relation[2]

$$T_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)dx, \quad \varphi \in \mathfrak{D}(\mathbb{R}).$$

Translating $f(x)$ by $\tau \in \mathbb{R}$, the function $f(x-\tau)$ defines a distribution $T_{f(x-\tau)}$ by

$$\begin{aligned} T_{f(x-\tau)}(\varphi) &= \int_{-\infty}^{\infty} f(x-\tau)\varphi(x)dx \\ &= \int_{-\infty}^{\infty} f(x)\varphi(x+\tau)dx, \quad \varphi \in \mathfrak{D}(\mathbb{R}). \end{aligned}$$

Hence, if f is a periodic function with period τ (or simply, τ- periodic), we must have

$$\int_{-\infty}^{\infty} f(x)\varphi(x)dx = \int_{-\infty}^{\infty} f(x)\varphi(x+\tau)dx$$

for all $\varphi \in \mathfrak{D}(\mathbb{R})$. Generalizing this reasoning, the concept of periodic distribution is defined as follows. [3]

Definition 2.1. A distribution $T \in \mathfrak{D}(\mathbb{R})'$ is called a **periodic distribution** with period τ if

$$T(\varphi(x)) = T(\varphi(x+\tau))$$

for all $\varphi \in \mathfrak{D}(\mathbb{R})$.

We denote by $\mathfrak{D}_\tau(\mathbb{R})'$ the set of τ- periodic distributions. For the sake of simplicity, we assume $\tau = 2\pi$ from now on.

To start with, let me confirm that $\mathfrak{D}_{2\pi}(\mathbb{R})'$ can be identified with $\mathfrak{C}^\infty(\mathbb{T}, \mathbb{C})'$.
Assume that $S \in \mathfrak{C}^\infty(\mathbb{T}, \mathbb{C})'$ is given. We associate to each $\varphi \in \mathfrak{D}(\mathbb{R})$ a new function

$$\tilde{\varphi}(x) = \sum_{n=-\infty}^{\infty} \varphi(x+2n\pi) \tag{2.1}$$

The value $\tilde{\varphi}(x)$ is defined without any ambiguity because the support of φ is compact and so the right-hand side of (2.1) is actually a finite sum for each x. Since $\tilde{\varphi}$ is 2π- periodic and infinitely differentiable, it can be regarded as an element of $\mathfrak{C}^\infty(\mathbb{T}, \mathbb{C})$. If we define an operator T on $\mathfrak{D}(\mathbb{R})$ by

$$T(\varphi) = S(\tilde{\varphi}), \quad \varphi \in \mathfrak{D}(\mathbb{R}). \tag{2.2}$$

[2]We denote by $\mathfrak{L}^1_{loc.}(\mathbb{R}, \mathbb{C})$ the space of locally integrable complex-valued functions defined on $\mathbb{R}$. $\mathfrak{D}(\mathbb{R})$ is the space of test functions.
[3]$\mathfrak{D}(\mathbb{R})'$ denotes the dual space of $\mathfrak{D}(\mathbb{R})$. Each element of $\mathfrak{D}(\mathbb{R})'$ is called a distribution.

T is a continuous linear operator and so $T \in \mathfrak{D}(\mathbb{R})'$. Since it is obvious that $T(\varphi(x)) = T(\varphi(x+2\pi))$, T is a 2π- periodic distribution; i.e. $T \in \mathfrak{D}_{2\pi}(\mathbb{R})'$.

Conversely, let T be an element of $\mathfrak{D}_{2\pi}(\mathbb{R})'$. It can be shown that $T \in \mathfrak{C}^{\infty}(\mathbb{T}, \mathbb{C})'$. We prepare a lemma due to Yosida-Kato[12], §7.

Lemma 2.1. For any $a > 0$, there exists some $\theta \in \mathfrak{D}(\mathbb{R})$ which satisfies[4]

(i) $\operatorname{supp} \theta = [-a, a]$, and

(ii) $\displaystyle\sum_{n=-\infty}^{\infty} \theta(x + na) = 1.$

Proof. Define a function $\theta : \mathbb{R} \to \mathbb{C}$ by

$$\theta(x) = \begin{cases} \displaystyle\int_{|x|}^{a} \exp\left(-\frac{1}{w(a-w)}\right) dw \Big/ \int_{0}^{a} \exp\left(-\frac{1}{w(a-w)}\right) dw & \text{for} \quad |x| \leqq a, \\ 0 & \text{for} \quad |x| > a. \end{cases}$$

Then it is clear that $\theta \in \mathfrak{D}(\mathbb{R})$ and (i) is satisfied. Computing $\theta(x-a)$ for $|x| \leqq a$, we obtain that

$$\theta(x-a) = \int_{|x-a|}^{a} \exp\left(-\frac{1}{w(a-w)}\right) dw \Big/ \int_{0}^{a} \exp\left(-\frac{1}{w(a-w)}\right) dw.$$

Changing the variable by $z = a - w$, we can rewrite it as

$$\begin{aligned} \theta(x-a) &= -\int_{x}^{0} \exp\left(-\frac{1}{z(a-z)}\right) dz \Big/ \int_{0}^{a} \exp\left(-\frac{1}{w(a-w)}\right) dw \\ &= \int_{0}^{x} \exp\left(-\frac{1}{z(a-z)}\right) dz \Big/ \int_{0}^{a} \exp\left(-\frac{1}{w(a-w)}\right) dw \end{aligned}$$

if $x \in [0, a]$. Consequently it follows that

$$\theta(x) + \theta(x-a) = 1 \quad \text{for} \quad x \in [0, a].$$

We also obtain the same relation for $x \in [-a, 0)$ by a similar argument.

There exists, for each $x \in \mathbb{R}$, a unique integer $k \in \mathbb{Z}$ such that

$$ka \leqq x < (k+1)a.$$

Therefore we must have

$$\sum_{n=-\infty}^{\infty} \theta(x + na) = \theta(x - ka) + \theta(x - (k+1)a) = 1.$$

This proves (ii). □

[4]supp θ means the support of the function θ.

Let us go back to prove that $T \in \mathfrak{D}_{2\pi}(\mathbb{R})'$ can be regarded as an element of $\mathfrak{C}^\infty(\mathbb{T},\mathbb{C})'$.

Any $\psi \in \mathfrak{C}^\infty(\mathbb{T},\mathbb{C})$ can be regarded as a 2π- periodic smooth function defined on $\mathbb{R}$. If $\theta \in \mathfrak{D}(\mathbb{R})$ is a function which satisfies (i) and (ii) of Lemma 1 for $a = \pi$, then $\psi\theta \in \mathfrak{D}(\mathbb{R})$. Define an operator U on $\mathfrak{C}^\infty(\mathbb{T},\mathbb{C})$ by

$$U(\psi) = T(\psi\theta). \tag{2.3}$$

It is well-defined in the sense that U does not depend upon the choice of θ. In fact, if $\eta \in \mathfrak{D}(\mathbb{R})$ satisfies

$$\sum_{n=-\infty}^{\infty} \eta(x + 2n\pi) = 0,$$

we have [5]

$$\begin{aligned} T(\psi\eta) &= T(\psi(x)\eta(x) \sum_{n=-\infty}^{\infty} \theta(x + 2n\pi)) \\ &= \sum_{n=-\infty}^{\infty} T(\psi(x - 2n\pi)\eta(x - 2n\pi)\theta(x)) \quad \text{(by the periodicity of } T\text{)} \\ &= \sum_{n=-\infty}^{\infty} T(\eta(x - 2n\pi) \cdot \psi(x)\theta(x)) \quad \text{(by the periodicity of } \psi\text{)} \\ &= T\left(\left(\sum_{n=-\infty}^{\infty} \eta(x - 2n\pi)\right)\psi(x)\theta(x)\right) \\ &= 0. \end{aligned}$$

This confirms that U is well-defined.

U is continuous on $\mathfrak{C}^\infty(\mathbb{T},\mathbb{C})$. In fact, for any net $\{\psi_\alpha\}$ in $\mathfrak{C}^\infty(\mathbb{T},\mathbb{C})$ which converges to some $\psi \in \mathfrak{C}^\infty(\mathbb{T},\mathbb{C})$, we have [6]

$$\psi_\alpha\theta \to \psi\theta \quad \text{in} \quad \mathfrak{D}(\mathbb{R}).$$

Consequently it follows that

$$U(\psi_\alpha) = T(\psi_\alpha\theta) \to T(\psi\theta) = U(\psi).$$

Thus we established the chain:

$$\begin{array}{ccccc} S & \xmapsto{(2.2)} & T & \xmapsto{(2.3)} & U. \\ \in \mathfrak{C}^\infty(\mathbb{T},\mathbb{C})' & & \in \mathfrak{D}_{2\pi}(\mathbb{R})' & & \in \mathfrak{C}^\infty(\mathbb{T},\mathbb{C})' \end{array}$$

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[5] $\sum_{k=-p}^{p} \theta(x + 2k\pi) \to \sum_{n=-\infty}^{\infty} \theta(x + 2n\pi)$ (in $\mathfrak{C}^\infty$) on supp $\psi\eta$.

[6] We should note that supp $\psi_\alpha\theta \subset$ supp θ.

It is a natural question to ask if S is identical with U. The answer is positive as confirmed by the following calculation:

$$\begin{aligned}U(\psi) &= T(\psi\theta)\\ &= S\left(\sum_{n=-\infty}^{\infty}\psi(x+2n\pi)\theta(x+2n\pi)\right)\\ &= S\left(\sum_{n=-\infty}^{\infty}\psi(x)\theta(x+2n\pi)\right) \quad \text{(by the periodicity of } \psi)\\ &= S\left(\psi(x)\sum_{n=-\infty}^{\infty}\theta(x+2n\pi)\right)\\ &= S(\psi) \quad \text{for any} \quad \psi\in\mathfrak{C}^\infty(\mathbb{T},\mathbb{C}).\end{aligned}$$

Summing up, we obtain the following result.

Theorem 2.1. *There is a one-to-one correspondence between* $\mathfrak{C}^\infty(\mathbb{T},\mathbb{C})'$ *and* $\mathfrak{D}_{2\pi}(\mathbb{R})'$.

The operators defined by (2.2) and (2.3) are the inverse operators to each other.

3. Fourier Coefficients of a Periodic Distribution

Let T be a 2π- periodic distribution; i.e. $T\in\mathfrak{D}_{2\pi}(\mathbb{R})'$. The Fourier coefficients of T are defined by

$$c_n = \frac{1}{\sqrt{2\pi}}T(e^{-inx}), \quad n\in\mathbb{Z}.$$

The formal series

$$\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

is called the Fourier series of T.

Remark. Assume that a trigonometric series

$$\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}C_n e^{inx} \tag{3.1}$$

simply converges to a distribution T; i.e.

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sum_{k=-p}^{p}C_k e^{ikx}\cdot\varphi(x)dx \to T(\varphi) \quad \text{for any} \quad \varphi\in\mathfrak{D}_{2\pi}(\mathbb{R}). \tag{3.2}$$

If we consider a special case where $\varphi = e^{-inx}$, the left-hand side of (3.2) converges to $\sqrt{2\pi}C_n$. On the other hand, $T(e^{-inx}) = \sqrt{2\pi}c_n$. Hence we must have $C_n = c_n$, and so the series (3.1) is nothing other than the Fourier series of T.

The following several facts are well-known in the theory of distributions.

1° If a sequence $\{T_n\} \in \mathfrak{D}(\mathbb{R})'$ simply converges to some $T \in \mathfrak{D}(\mathbb{R})'$, then the sequence $\{T_n'\}$ of the derivatives (in the sense of distribution) also simply converges to T'. (More generally, the sequence $\{D^p T_n\}$ of the p-th derivatives simply converges to $D^p T$.)

2° A sequence $\{T_n\}$ in $\mathfrak{S}(\mathbb{R})'$ (the space of tempered distributions) [7] simply converges to $T \in \mathfrak{S}(\mathbb{R})'$ if and only if the sequence $\{\hat{T}_n\}$ of the Fourier transforms simply converges to $\hat{T}$.

3° The Fourier transforms of δ and e^{inx} are computed as follows:

$$\hat{\delta} : \varphi \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x)dx, \quad \varphi \in \mathfrak{S}(\mathbb{R}). \quad \widehat{e^{inx}} = \sqrt{2\pi}\delta_n.$$

4°

$$\begin{aligned} \sum_{k=-n}^{n} c_k \delta_k \text{ simply converges} &\Longleftrightarrow \sum_{k=-n}^{n} c_k \widehat{e^{ikx}} \text{ simply converges} \\ &\Longleftrightarrow \sum_{k=-n}^{n} c_k e^{ikx} \text{ simply converges.} \end{aligned}$$

These relations can be confirmed by combining 2° and 3°.

Theorem 3.1. *Consider a sequence $\{c_n\}_{n\in\mathbb{Z}}$ of complex numbers. c_n's are the Fourier coefficients of some 2π-periodic distribution if and only if there exists some $N \in \mathbb{N} \cup \{0\}$ such that*

$$\begin{gathered} c_n = O(|n|^N); \\ \text{i.e.} \quad |c_n| \leqq K|n|^N \quad \text{for some} \quad K > 0. \end{gathered} \tag{3.3}$$

Proof. [8] Assume first that $\{c_n\}$ satisfies (3.3). We write formally

$$u(x) = \sum_{n\neq 0} \frac{1}{(in)^{N+2}} c_n e^{inx}, \quad x \in \mathbb{R}. \tag{3.4}$$

Since

$$\sum_{n\neq 0} \frac{1}{|n|^{N+2}} |c_n||e^{inx}| \leqq \sum_{n\neq 0} \frac{K}{n^2}$$

by the assumption (3.3), the right-hand side of (3.4) is absolutely, uniformly convergent. Hence $u(x)$ is a continuous function which satisfies

$$|u(x)| \leqq K \sum_{n\neq 0} \frac{1}{n^2}. \tag{3.5}$$

[7] $\mathfrak{S}(\mathbb{R})$ denotes the space of rapidly decreasing functions. $\mathfrak{S}(\mathbb{R})'$ is its dual space, each element of which is called a tempered distribution.

[8] See Folland[4], pp.320-322 and Lax[7], p.570 for outline.

Consequently, $u(x)$ defines a distribution, and

$$u_n(x) = \sum_{\substack{k=-n \\ k\neq 0}}^{n} \frac{1}{(ik)^{N+2}} c_k e^{ikx}$$

simply converges to $u(x)$ $\big($exactly speaking, to the distribution defined by $u_n(x)\big)$. In fact, it can be verified by

$$\left| \int_{-\infty}^{\infty} (u_n(x) - u(x))\varphi(x)dx \right| \leqq \int_{-\infty}^{\infty} |u_n(x) - u(x)| \cdot |\varphi(x)| dx$$
$$\to 0 \quad \text{as} \quad n \to \infty \quad \text{for any} \quad \varphi \in \mathfrak{D}(\mathbb{R}).$$

It follows that, taking account of 1°,

$$D^{N+2}u(x)(\varphi) = \lim_{n\to\infty} \sum_{\substack{k=-n \\ k\neq 0}}^{n} \frac{1}{(ik)^{N+2}} c_k (ik)^{N+2} e^{ikx}(\varphi) = \lim_{n\to\infty} \sum_{\substack{k=-n \\ k\neq 0}}^{n} c_k e^{ikx}(\varphi).$$

Hence we must have

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} c_k e^{ikx} \to \frac{1}{\sqrt{2\pi}}(c_0 + D^{N+2}u(x)) \quad \text{simply as} \quad n \to \infty.$$

By the remark stated at the beginning of this section, c_n's are the Fourier coefficient of the 2π- periodic distribution defined by $(\frac{1}{\sqrt{2\pi}})(c_0 + D^{N+2}u(x))$.

Let us go over to the converse. Assume that

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} c_k e^{ikx}$$

simply converges to some distribution. If c_n's do not satisfy (3.3), there exists a sequence $\{n_r\}$ of integers such that

$$|c_{n_r}| > |n_r|^r, \quad r = 1, 2, \cdots \tag{3.6}$$

Define a couple of functions $\lambda(x)$ and $\varphi(x)$ by

$$\lambda(x) = \begin{cases} e^{-x^2/(1-x^2)} & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geqq 1 \end{cases} \tag{3.7}$$

and

$$\varphi(x) = \sum_{r=1}^{\infty} c_{n_r}^{-1} \lambda(x - n_r). \tag{3.8}$$

The right-hand side of (3.8) is a finite sum for each x and $\varphi(x) \in \mathfrak{D}(\mathbb{R})$. By definition of φ, we have[9]

$$\varphi(n) = \begin{cases} 0 & \text{if} \quad n \neq n_r, \\ c_{n_r}^{-1} & \text{if} \quad n = n_r \end{cases}$$

[9] $n \neq n_r \Rightarrow \lambda(n - n_r) = 0, \;\; n = n_r \Rightarrow \lambda(n_r - n_r) = 1.$

$$(r = 1, 2, \cdots).$$

It follows that

$$\begin{aligned}\sum_{k=-n}^{n} c_k \delta_k(\varphi) &= \int_{-\infty}^{\infty} \left\{ \sum_{k=-n}^{n} c_k \delta_k \right\} \varphi(t) dt \\ &= \sum_{k=-n}^{n} c_k \varphi(k) \qquad (3.9) \\ &= \text{the number of } {n_r}'s \text{ between } -n \text{ and } n.\end{aligned}$$

The right-hand side of (3.9) diverges to ∞ as $n \to \infty$. Consequently, by 4°, $\sum_{k=-n}^{n} c_k e^{ikt}$ can not be simply convergent. Contradiction. □

4. Herglotz's Theorem

Definition 4.1. Let $(G, +)$ be a commutative group. A function $f : G \to \mathbb{C}$ is **positive semi-definite** if

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j f(x_i - x_j) \geqq 0$$

for any finite elements $x_1, x_2, \cdots, x_n$ of G and any complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$.

A sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers can be regarded as a function of $G = \mathbb{Z}$ into $\mathbb{C}$. Hence $\{a_n\}_{n\in\mathbb{Z}}$ is said to be positive semi-definite if

$$\sum_{i,j}^{n} \lambda_i \bar{\lambda}_j a_{i-j} \geqq 0$$

for any finitely many complex numbers $\{\lambda_i\}$.

The following elementary properties of positive semi-definite functions are well-known.

1° $f(0) \geqq 0$.
2° $\overline{f(x)} = f(-x)$.
3° $|f(x)| \leqq f(0)$.
4° If $G = \mathbb{R}$ and f is continuous at 0, then f is uniformly continuous on $\mathbb{R}$.
5° In case of $G = \mathbb{R}$, f is positive semi-definite if and only if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)\theta(x)\overline{\theta(y)} dx dy \geqq 0$$

for any $\theta \in \mathfrak{D}(\mathbb{R})$.

We now state and prove the theorem due to Herglotz[5], which characterizes the Fourier transform of a positive Radon measure on $\mathbb{T}$ as a positive semi-definite numerical sequence.[10]

[10] A similar result was obtained by Carathéodory[3].

A Radon measure μ on $\mathbb{T}$ is a 2π- periodic distribution (i.e. $\mu \in \mathfrak{D}_{2\pi}(\mathbb{R})' \cong \mathfrak{C}^\infty(\mathbb{T}, \mathbb{C})'$). Hence its Fourier coefficients are defined by

$$a_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-in\theta} d\mu(\theta), \quad n \in \mathbb{Z}. \tag{4.1}$$

Theorem 4.1 (Herglotz)**.** *The following two statements are equivalent for a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers.*

(i) $\{a_n\}_{n\in\mathbb{Z}}$ *is positive semi-definite.*

(ii) a_n*'s are the Fourier transforms of some positive Radon measure μ on $\mathbb{T}$.*

Proof. [11] The proof of "if" part is easy and well-known. So it is enough to show "only if" part.

Assume that $\{a_n\}_{n\in\mathbb{Z}}$ is positive semi-definite. By the property 3° above, we have

$$|a_n| \leqq a_0 \quad \text{for all} \quad n \in \mathbb{Z}. \tag{4.2}$$

Hence it is obvious that $|a_n| \leqq$ const. $|n|^N$ for some $N \in \mathbb{N} \cup \{0\}$. It follows from Theorem 3.1 that a_n's are the Fourier coefficients of some 2π- periodic distribution T; i.e.

$$a_n = \frac{1}{\sqrt{2\pi}} T(e^{-inx}). \tag{4.3}$$

Let φ be any element of $\mathfrak{D}_{2\pi}(\mathbb{R})$ and φ_k's its Fourier coefficients. By a simple computation, we have

$$T\varphi = T\left(\frac{1}{\sqrt{2\pi}} \sum \varphi_n e^{inx}\right) = \frac{1}{\sqrt{2\pi}} \sum \varphi_n T(e^{inx}) = \sum \bar{a_n} \varphi_n. \tag{4.4}$$

Since φ is 2π-periodic and smooth, the series summing its Fourier coefficients is absolutely convergent.[12] Taking account of (4.2), we observe that the right-hand side of (4.4) converges.

We now proceed to show that T is positive.

Let $q_N(x)$ be any trigonometric polynomial of order N;

$$q_N(x) = \sum_{n=-N}^{N} \phi_n e^{inx} \tag{4.5}$$

If we adopt

$$|q_N(x)|^2 = \sum_{n,k=-N}^{N} \phi_n \bar{\phi}_k e^{i(n-k)x}.$$

as $\varphi \in \mathfrak{D}_{2\pi}(\mathbb{R})$, (4.4) implies

$$T(|q_N(x)|^2) = \sqrt{2\pi} \sum a_{k-n} \phi_n \bar{\phi}_k \geqq 0. \tag{4.6}$$

[11] I appreciate the priority of Lax[7], pp.142-143 for the ideas of proof.

[12] cf. Katznelson[6], p.26.

Any $q(x) \in \mathfrak{D}_{2\pi}(\mathbb{R})$ can be approximated by a sequence of trigonometric polynomials in $\mathfrak{C}^\infty$- topology. Passing to the limit, (4.6) implies

$$T(|q(x)|^2) \geqq 0 \quad \text{for any} \quad q(x) \in \mathfrak{D}_{2\pi}(\mathbb{R}). \tag{4.7}$$

Let $p(x) \in \mathfrak{D}_{2\pi}(\mathbb{R})$ be non-negative (real-valued) and $q(x) = \sqrt{p(x)}$. Then $q(x) \in \mathfrak{D}_{2\pi}(\mathbb{R})$ and

$$T(p(x)) \geqq 0$$

by (4.7). Consequently, the distribution T is positive and hence it is a positive measure;[13] i.e.

$$T(\varphi) = \int_{\mathbb{T}} \varphi(x) d\mu \quad \text{for all} \quad \varphi \in \mathfrak{C}^\infty(\mathbb{T}, \mathbb{C})$$

for some $\mu \in \mathfrak{M}_+(\mathbb{T})$. So we must have the desired result:

$$a_n = \frac{1}{\sqrt{2\pi}} T(e^{-inx}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-inx} d\mu.$$

□

5. Bochner's Theorem

In this section, we discuss the celebrated theorem due to Bochner[1], which is an analogous version of Herglotz's Theorem in the case $G = \mathbb{R}$ rather than $G = \mathbb{Z}$.

Remark. Since $\mathfrak{S}(\mathbb{R})$ is a dense subspace of $\mathfrak{C}_\infty(\mathbb{R}, \mathbb{C})$ (space of continuous functions vanishing at infinity), any $\mu \in \mathfrak{M}$(space of Radon measures on $\mathbb{R}$) can be regarded as a tempered distribution; i.e.$\mu \in \mathfrak{S}(\mathbb{R})'$. Here arises a question. Is the usual Fourier transform (Fourier-Stieltjes transform) of μ coincide with its Fourier transform in the sense of distribution? Let us evaluate the Fourier transform $\hat{\mu}(\theta)$ in the sense of distribution for $\theta \in \mathfrak{S}(\mathbb{R})$.

$$\begin{aligned}
\hat{\mu}(\theta) = \mu(\hat{\theta}) &= \mu\left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \theta(x) e^{-i\xi x} dx\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \theta(x) e^{-i\xi x} dx\right] d\mu(\xi) \\
&= \int_{\mathbb{R}} \theta(x) \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} d\mu(\xi)\right] dx \\
&= \int_{\mathbb{R}} \theta(x) \hat{\mu}(\xi) dx.
\end{aligned}$$

($\hat{\mu}(\xi)$is the usual Fourier transform of μ.)

This equality holds true for any $\theta \in \mathfrak{S}(\mathbb{R})$. Consequently we conclude that the two definitions are identical.

[13]See Schwartz[11], Chap.1, §4, Théorème 5. $\mathfrak{M}_+(\mathbb{T})$ denotes the set of all the positive Radon measures on $\mathbb{T}$.

Theorem 5.1 (Bochner). *The following two statements are equivalent for a function φ : $\mathbb{R} \to \mathbb{C}$.*

(i) *φ is positive semi-definite and continuous at 0.*[14]

(ii) *φ can be represented as the Fourier transform of a positive Radon measure on $\mathbb{R}$.*

Proof. [15] As in Theorem 4.1, (ii)⇒(i) is easy and well-known. So we have only to prove (i)⇒(ii).

Assume(i). Then φ is bounded since

$$|\varphi(x)| \leqq \varphi(0) \quad \text{for all} \quad x \in \mathbb{R} \tag{5.1}$$

by the property 3° of positive semi-definite functions. Hence φ defines a tempered distribution ($\in \mathfrak{S}(\mathbb{R}')$). We denote by $\check{\varphi} \in \mathfrak{S}(\mathbb{R})'$ the inverse Fourier transform (as a distribution). By Parseval's theorem, we have [16]

$$\varphi(s) = (\check{\varphi})\hat{}(s) = \check{\varphi}(\hat{s}), \quad \text{for} \quad s \in \mathfrak{S}(\mathbb{R}). \tag{5.2}$$

The positive semi-definiteness of φ implies (by the property 5°)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x-y)\theta(x)\overline{\theta(y)}dxdy \geqq 0 \quad \text{for any} \quad \theta \in \mathfrak{D}(\mathbb{R}),$$

which can be rewritten as (changing the variable $z = x - y$)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(z)\theta(x)\overline{\theta(x-z)}dxdz \geqq 0. \tag{5.3}$$

If we define the function $\Theta \in \mathfrak{D}(\mathbb{R})$ by

$$\Theta(z) = \int_{\mathbb{R}} \theta(x)\overline{\theta(x-z)}dx, \tag{5.4}$$

(5.3) can be rewritten as

$$\int_{\mathbb{R}} \varphi(z)\Theta(z)dz \geqq 0. \tag{5.3'}$$

[14] By the property 4° of positive semi-definite functions, φ is uniformly continuous on $\mathbb{R}$ if it is continuous at 0.

[15] The basic ideas are given in Lax [7], pp.144-146.

[16] Here φ denotes the tempered distribution defined by the bounded function φ. Hence

$$\varphi(s) = \int_{\mathbb{R}} \varphi(x)s(x)dx, \quad s \in \mathfrak{S}(\mathbb{R}).$$

By (5.4),

$$\begin{aligned}
\hat{\Theta}(w) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \theta(x)\overline{\theta(x-z)}dx \right] e^{-iwz}dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \theta(x)\overline{\theta(u)}dx \right] e^{-iwx}e^{iwu}du \quad \text{(changing the variable } u = x - z) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \theta(x)e^{-iwx}dx \cdot \int_{\mathbb{R}} \bar{\theta}(u)e^{iwu}du \\
&= \hat{\theta}(w) \cdot \sqrt{2\pi}\bar{\hat{\theta}}(w) \\
&= \sqrt{2\pi}|\hat{\theta}(w)|^2.
\end{aligned}$$

It follows that

$$|\hat{\theta}(w)|^2 = \hat{\theta}(w)\bar{\hat{\theta}}(w) = \frac{1}{\sqrt{2\pi}}\hat{\Theta}(w). \tag{5.5}$$

(5.2), (5.3') and (5.5) imply

$$\varphi(\Theta) \underset{(5.2)}{=} \check{\varphi}(\hat{\Theta}) \underset{(5.5)}{=} \check{\varphi}(\sqrt{2\pi}|\hat{\theta}|^2) \underset{(5.3')}{\geqq} 0. \tag{5.6}$$

The inequality (5.6) holds true for any $\theta \in \mathfrak{D}(\mathbb{R})$. Since $\mathfrak{D}(\mathbb{R})$ is dense in $\mathfrak{S}(\mathbb{R})$, (5.6) is also valid for any $\theta \in \mathfrak{S}(\mathbb{R})$.

Let $p(w) \geqq 0$ be an element of $\mathfrak{C}_0^\infty(\mathbb{R}, \mathbb{R})$(space of smooth functions with compact support). Then $\sqrt{p(w)}$ is also an element of $\mathfrak{C}_0^\infty(\mathbb{R}, \mathbb{R}) \subset \mathfrak{S}(\mathbb{R})$. Using $\sqrt{p}$ as $|\hat{\theta}|$ in (5.6), we obtain [17]

$$\check{\varphi}(p) \geqq 0 \quad \text{for any} \quad 0 \leqq p \in \mathfrak{C}_0^\infty(\mathbb{R}, \mathbb{R}). \tag{5.7}$$

(5.7) tells us that $\check{\varphi}$ is a positive distribution, and so it is a positive measure; i.e.

$$\check{\varphi}(\theta) = \int_{\mathbb{R}} \theta(x)d\mu, \quad \theta \in \mathfrak{D}(\mathbb{R}) \tag{5.8}$$

for some positive measure μ.

Finally we claim that $\mu(\mathbb{R}) < \infty$. Let g be an element of $\mathfrak{C}_0^\infty(\mathbb{R}, \mathbb{R})$ which is non-negative and satisfies

$$g(x) = 1 \quad \text{on} \quad [-1, 1]. \tag{5.9}$$

Define a function g_n by $g_n(x) = g(x/n)$. Let G and G_n be the inverse Fourier transforms of g and g_n, respectively. Taking account of $G_n(y) = nG(ny)$, we apply (5.6) to get

$$\check{\varphi}(g_n) = \int_{\mathbb{R}} \varphi(y)nG(ny)dy. \tag{5.10}$$

[17] We note that there exists a unique function in $\mathfrak{S}(\mathbb{R})$, the Fourier transform of which is just $\sqrt{p}$.

Since $\check{\varphi}$ is a positive distribution, the properties of g implies

$$\check{\varphi}(g_n) \geqq \int_{[-n,n]} d\mu.$$

Furthermore, it follows from (5.1) that

$$\int_{\mathbb{R}} \varphi(y) nG(ny) \leqq \varphi(0) \int_{\mathbb{R}} n|G(ny)|dy = \varphi(0) \int_{\mathbb{R}} |G(y)|dy. \tag{5.11}$$

The right-hand side of (5.11) is independent of n. If we write

$$C = \varphi(0) \int_{\mathbb{R}} |G(y)|dy,$$

we obtain

$$\int_{[-n,n]} d\mu \leqq C,$$

and so $\mu(\mathbb{R}) \leqq C$.

Looking at (5.8), we can conclude that φ is the Fourier transform of the distribution defined by μ; i.e.

$$\varphi(s) = \hat{\mu}(s) \quad \text{for} \quad s \in \mathfrak{D}(\mathbb{R}).$$

□

6. An Application

The main purpose of this article is to shed a new light on the mathematical structure of the Herglotz-Bochner theorem from the viewpoint of Schwartz's distribution theory. However, in this final section, I dare make a digression to exemplify a typical way of its application. Among numerous candidates, I would like to choose the problem to characterize the periodicity (or almost-periodicity) of a weakly stationary stochastic process. It is well-known that this problem plays a prominent role in time-series analysis, mathematical theory of business cycles, and so on. I try to describe its outline as briefly as possible, keeping in mind that this section is just a superfluous digression. Readers can find a systematic discussion (as well as a detailed bibliography) of the problem in my recent article Maruyama [9].

Let $(\Omega, \mathcal{E}, \mathcal{P})$ be a probability space. $X(t,\omega) : \mathbb{R} \times \Omega \to \mathbb{C}$ is assumed to be a weakly stationary stochastic process which is $(\mathcal{L} \otimes \mathcal{E}, \mathcal{B}(\mathbb{C}))$- measurable, where $\mathcal{L}$ is the Lebesgue σ-field on $\mathbb{R}$. Then it is not difficult to show that the covariance function $\rho(u)$ of $X(t,\omega)$ is positive semi-definite. Thanks to the Bochner theorem, $\rho(u)$ can be expressed as the Fourier transform of certain positive Radon measure ν on $\mathbb{R}$: i.e.

$$\rho(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iut} d\nu(t).$$

Such a Radon measure ν is determined uniquely and is called the spectral measure of $X(t,\omega)$.

We can characterize the periodicity of $X(t,\omega)$ by showing the equivalence of the following three statements.

(i) $\rho(u)$ is a periodic function with period τ.
(ii) $X(t+\tau,\omega) - X(t,\omega) = 0 \quad a.e.(\omega)$ for all $t \in \mathbb{R}$.
(iii) If $E \in \mathcal{B}(\mathbb{R})$ and $E \cap \{2k\pi/\tau \mid k \in \mathbb{Z}\} = \emptyset$, then $\nu(E) = 0$.

If any one of the three statements holds true, the spectral measure concentrates on a countable set in $\mathbb{R}$ such that the distance of any adjacent two points is some multiple of $2\pi/\tau$.

We obtain a similar result for a weakly stationary stochastic process $X(n,\omega) : \mathbb{Z}\times\Omega \to \mathbb{C}$ with discrete time. In this case, we should make use of the Herglotz theorem rather than the Bochner theorem. As to the almost-periodicity, see also Maruyama [9].

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