Journal of the Operations Research Society of Japan
Vol. 60, No. 2, April 2017, pp. 136–155

SUPER-STRONG REPRESENTATION THEOREMS FOR NONDETERMINISTIC SEQUENTIAL DECISION PROCESSES

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(Received March 1, 2016; Revised November 16, 2016)

Abstract This paper studies the relation between a given nondeterministic discrete decision process (nd-ddp) and a nondeterministic sequential decision process (nd-sdp), which is a finite nondeterministic automaton with a cost function, and its subclasses (nd-msdp, nd-pmsdp, nd-smsdp). We show super-strong representation theorems for nd-sdp and its subclasses, for which the functional equations of nondeterministic dynamic programming are obtainable. The super-strong representation theorems provide necessary and sufficient conditions for the existence of the nd-sdp and its subclasses with the same set of feasible policies and the same cost value for every feasible policy as the given process nd-ddp.

Keywords: Dynamic programming, nondeterministic automaton, nondeterministic sequential decision process, super-strong representation theorem

1. Introduction

By using automata theory, Karp and Held [4] and Ibaraki [1] derived the relation between a given discrete decision process (ddp) and sequential decision process (sdp), and its subclasses, namely monotone sdp (msdp) and strictly monotone sdp (smsdp). For the process msdp, the functional equations of regular dynamic programming are obtainable. Ibaraki [1] also proved the relation between ddp and subclasses of msdp's for which simpler solution methods are available; in particular, he defined a sequential decision process named positively msdp, shortly, pmsdp. The process pmsdp is important because the Dijkstra's algorithm can be applied to the process. Moreover, Ibaraki [2] introduced a nondeterministic sdp (nd-sdp) and its subclasses, and investigated properties of the sets accepted by nd-sdp and its subclasses, where nondeterministic finite automaton M in nd-sdp accepts input string x if x sends M into one of the final states and the resulting cost $\bar{h}$ is not greater than a given threshold θ.

Further, Maruyama [6, 7] defined a bitone sequential decision process (bsdp) and subclasses (strictly bitone sdp (sbsdp) and loop-free bitone sdp (lbsdp)) of bsdp's. The bsdp admits a system of functional equations in bynamic programming proposed by Iwamoto [3] and contains the class of msdp's as a special case. In [8], he has also introduced an associative sequential decision process (assdp) whose objective function is defined through associative binary operations; the process assdp is a subclass of bsdp, which has the simplest structure of all bsdp. By using automata theory, he also made clear the relation between a given ddp and the process assdp.

In this paper, we will consider a different nd-sdp and its subclasses from those in Ibaraki [2], which correspond to nd-msdp, nd-pmsdp, nd-smsdp; but objective functions in this paper are of min-Max type. We will show super-strong representation theorems for the processes, which are necessary and sufficient conditions for the existence of the nd-sdp and its subclasses, that is, nd-msdp, nd-smsdp, nd-pmsdp with the same set of feasible policies and the same cost value for every feasible policy as the given process nd-ddp. In Section 2, the process nd-sdp and its subclasses are defined. In Section 3, we will prove a super-strong representation theorem for the nd-sdp on which super-strong representation theorems for its subclasses are based. Further, nondeterministic shortest path problem will be considered as a concrete example of nd-sdp. In Section 4, super-strong representation theorems for nd-msdp and nd-pmsdp will be shown by using some partial ordering and a directed graph. An egg-dropping problem will be discussed as an example of nd-pmsdp. In Section 5, we will give super-strong representation theorem for nd-smsdp and nondeterministic assdp.

2. Definitions

A **nondeterministic discrete decision process** (**nd-ddp**) $\Upsilon_{\min}$ is defined by a system $(\Sigma, S, f, \min)$, where,

Σ : a finite nonempty alphabet (a set of primitive decisions);
Σ^* : the set of all strings (policies) composed of symbols of Σ;
I : a finite set of indices; I^* : the set of all sequences of indices of I;
$\Sigma^* \ni \epsilon$: the null string; $I^* \ni \mu$: the null index;
$\Sigma^* \supset S$: the set of feasible policies, defined by
$S = \{x \in \Sigma^* \mid \exists i \in A(x) \text{ s.t. } \pi(i) \in A_F\}$, where
$I^* \supset A(x = a_1 \cdots a_n)$: the set of indicies for a given x,
satisfying that $i_x \in A(x),\ i_z \in A(z) \implies i_x i_z \in A(xz)$,
$\pi(i) = i_n$:final index of i, A_F : the set of final indices;
$f_i(x) \in R^1$: defined for each x and index $i \in A(x)$, that is,
$\Sigma^* \ni x \to f_{A(x)}(x) = \{f_i(x) \mid i \in A(x)\}$: set-valued function,
$f : S \longrightarrow R^1 \cup \{\infty\}$: the cost function which is minimized:

$$f(x) = \begin{cases} \mathrm{Max}\{f_i(x) \mid i \in \bar{A}(x) = A(x)\} & ,\text{if } \bar{A}(x) = A(x), \\ \infty & ,\text{if } \bar{A}(x) \neq A(x) \end{cases}$$

$\Longrightarrow$ minimize for $x \in S$, where
$\bar{A}(x) = \{i \in A(x) \mid \pi(i) \in A_F\} \subset A(x)$.

A **nondeterministic finite automaton** (**nd-fa**)M is defined by a system $(Q, \Sigma, q_0, ST, Q_F)$, where Σ is the same as defined above, and

Q : a finite nonempty set of states; $Q \ni q_0$: an initial state;
$Q \times Q \times \Sigma \supset ST$: permitted state transitions, i.e., $(q, r, a) \in ST$ if and only if after taking policy $a \in \Sigma$, state transition from $q \in Q$ to $r \in Q$ is permitted;
$Q \supset Q_F$: the set of final states.

We note that after taking a policy a for a state q, $(q, r, a) \in ST$ means that the next state is not only one but some states r can be permitted.

Further, **nondeterministic sequential decision process** (**nd-sdp**) is a nondeterministic finite automaton with objective function and defined as follows: $\Pi_{\min} = (M, h, \xi_0, \min)$, where

$M = (Q, \Sigma, q_0, ST, Q_F)$: **nd-fa**;

$h : R^1 \times ST \to R^1$: a cost function, i.e., $h(\xi, q, r, a)$ is the cost value at r after the state transition $q \to r$ by taking policy a for $(q, r, a) \in ST$ and cost ξ at q;

$R^1 \ni \xi_0$: initial cost of initial state q_0; $\bar{h}_{q0;\mu}(\epsilon) = \xi_{q_0}$, μ denotes the path of length 0,

$\bar{h}_{q0;\sigma r}(xa) = h(\bar{h}_{q0;\sigma}(x), \pi(\sigma), r, a)$, $\sigma \in Y(q_0, x), (\pi(\sigma), r, a) \in ST$ $(\sigma r \in Y(q_0, xa))$,

where, $\pi(\sigma)$:the final state of path σ, and

$Y(q_0, x) = \{r_1 r_2 \dots r_k \mid (q_0, r_1, a_1) \in ST, (r_1, r_2, a_2) \in ST, \dots (r_{k-1}, r_k, a_k) \in ST\}$:

the set of sequence of states generated by $x = a_1 a_2 \cdots a_k$ applied to q_0;

$\bar{h}_{q_0} : \Sigma^* \to R^1 \cup \{\infty\}$: the cost function which is minimized:

$$\bar{h}_{q_0}(x) = \begin{cases} \text{Max} \ \ \bar{h}_{q_0;Y(q_0,x)}(x) = \text{Max}\{\bar{h}_{q_0;\sigma}(x) \mid \sigma \in Y(q_0, x), \pi(\sigma) \in Q_F\}, \\ \qquad \ \ \text{if } Y(q_0, x) = \bar{Y}(q_0, x), \\ \ \ \infty, \ \ \text{otherwise.} \end{cases}$$

$\Longrightarrow$ minimize for $x = a_1 a_2 \cdots a_k$, where

$\bar{Y}(q_0, x) = \{\sigma \in Y(q_0, x) \mid \pi(\sigma) \in Q_F\}$.

We denote by $F(M)$ the set of strings accepted by the nd-fa M, namely, $F(M) = \{x \in \Sigma^* | \exists \sigma \in Y(q_0, x) \text{ s.t. } \pi(\sigma) \in Q_F\}$. Further, the set of all feasible policies of $\Pi_{\min}$ is denoted by $F(\Pi_{\min})(= F(M))$.

Next, let us introduce subclasses of nd-sdp. Let $\Pi_{\min}$ be an nd-sdp. If h satisfies the monotonicity condition:

$$\xi_1 \leq \xi_2 \Longrightarrow h(\xi_1, q, r, a) \leq h(\xi_2, q, r, a) \ \text{ for } \forall (q, r, a) \in ST,$$

then, $\Pi_{\min}$ is called a *monotone* nd-sdp(**nd-msdp**).
An nd-sdp $\Pi_{\min}$ is called a *strictly monotone* nd-sdp(**nd-smsdp**) if

$$\xi_1 < \xi_2 \Longrightarrow h(\xi_1, q, r, a) < h(\xi_2, q, r, a) \ \text{ for } \forall (q, r, a) \in ST.$$

An nd-msdp $\Pi_{\min}$ is called a *positively monotone* nd-sdp(**nd-pmsdp**) if

$$h(\xi, q, r, a) \geq \xi \ \text{ for } \forall \xi \in R^1, \forall (q, r, a) \in ST.$$

An nd-msdp $\Pi_{\min}$ is called a *loop-free* nd-msdp(**nd-lmsdp**) if

$$|F(\Pi_{\min})| < \infty \text{ (that is, } F(\Pi_{\min}) \text{ is a finite set).}$$

These subclasses were introduced by Ibaraki [2], where the definition of $\Pi_{\min}$ in this paper is slightly different from those of Ibaraki [2].

Further, let us introduce a new subclass of **nd-smsdp** which is called *associative* nd-sdp (**nd-assdp**).

Definition 2.1 (associative nondeterministic sequential decision process)**.** Let $\Pi_{\min}$ be a nd-sdp:$(M, h, \xi_0, \min)$. We call $\Pi_{\min}$ an *associative* nd-sdp, if $h(\xi, q, a) = \xi \circ \psi(q, r, a)$, where the binary operation $\circ$ satisfies the following:

(i) $(A, \circ)$ is a semi group: $\circ : A \times A \longrightarrow A$, where $A \subset R^1$, and it satisfies the associative law, that is, $(a \circ b) \circ c = a \circ (b \circ c) \ \ \forall a, b, c \in A$;

(ii) there exists an unit element $e(\circ) \in A$, that is, $a \circ e(\circ) = e(\circ) \circ a = a \quad \forall\, a \in A$;

(iii) there exists an inverse element a^{-1} for each $a \in A$, that is, $a \circ a^{-1} = a^{-1} \circ a = e(\circ)$;

(iv) the binary operation satisfies the commutative law, that is, $a \circ b = b \circ a \quad \forall a, b \in A$;

(v) the binary operation satisfies the strict monotonicity,
that is, $a_1,\ a_2 \in A,\ a_1 < a_2 \Longrightarrow\ a \circ a_1 < a \circ a_2\ \forall a \in A$.

Example 2.1 (additive process). $\circ = +,\ A = R^1,\ e(\circ) = 0,\ a^{-1} = -a\ \ (\ a \in R^1)$.
Example 2.2 (multiplicative process). $\circ = \times,\ A = \{a \mid a > 0\},\ e(\circ) = 1$, $a^{-1} = \dfrac{1}{a}\ (\ a \neq 0\ )$.
Example 2.3 (multiplicative additive process). $a \circ b = a + b - ab, \quad A = \{a \mid a < 1\}$, $e(\circ) = 0,\ \ a^{-1} = \dfrac{a}{a-1}\ (\ a \neq 1\ )$.
Example 2.4 (fractional process). $a \circ b = \dfrac{a+b}{1+ab}, \quad A = (-1, 1),\ e(\circ) = 0$, $a^{-1} = -a\ (\ a \in (-1, 1)\ )$.

Let $\Pi_{\min}$ be an **nd-assdp** and let

$$F(p) = \min_{x \in S} \left\{ \mathrm{Max}[\bar{h}_{p;\sigma}(x) \mid \sigma \in Y(p;x), \pi(\sigma) \in Q_F] \right\}$$

for $p \in Q$. Then we have

$$\begin{aligned} F(p) &= \min_{a \in \Sigma} \{\mathrm{Max}[\psi(p,q,a) \circ F(q) | (p,q,a) \in ST\} \quad \text{if } p \notin Q_F, \\ F(p) &= 0 \quad \text{if } p \in Q_F. \end{aligned}$$

These are the recursive functional equations of nondeterministic dynamic programming (see [5]).

Let us consider an nd-ddp: $\Upsilon_{\min} = (\Sigma, S, f, \min)$, $f(x) = \mathrm{Max} f_{\bar{A}(x)}(x)$ and an nd-sdp: $\Pi_{\min} = (M,\ h, \xi_0, \min)$. Then $\Pi_{\min}$ *super-strongly represents* $\Upsilon_{\min}$ if

$$\begin{aligned} & F(\Pi_{\min}) = S,\ \ \bar{Y}(q_0, x) \equiv \bar{A}(x) \ (\text{i.e. } \delta_x \in \bar{Y}(q_0, x) \Longleftrightarrow i_x \in \bar{A}(x))\ \forall x \in S, \\ & \bar{h}_{q_0;\delta_x}(x) = f_{i_x}(x) \quad \forall x, \forall \delta_x, \forall i_x;\ (x, \delta_x) \in \bar{F}(\Pi_{\min}),\ (x, i_x) \in S_{\bar{A}}, \\ & \text{where } \bar{F}(\Pi_{\min}) = \bigcup_{x \in F(\Pi_{\min})} \{(x, \delta) \mid \delta \in \bar{Y}(q_0, x)\},\quad S_{\bar{A}} = \bigcup_{x \in S} \{(x, i) \mid i \in \bar{A}(x)\}. \end{aligned}$$

Next, $\Pi_{\min}$ *strongly represents* $\Upsilon_{\min}$ if

$$F(\Pi_{\min}) = S,\ \bar{h}(x) = \mathrm{Max}\ \bar{h}_{q_0;\bar{Y}(q_0,x)}(x) = \mathrm{Max} f_{\bar{A}(x)}(x) = f(x)\ \forall x \in S$$

hold. Finally, $\Pi_{\min}$ *weakly represents* $\Upsilon_{\min}$ if

$$\begin{aligned} O(\Pi_{\min}) &= \{x \in F(\Pi_{\min}) \mid \bar{h}(x) \leq \bar{h}(y)\ \forall y \in F(\Pi_{\min})\} \\ &= \{x \in S \mid f(x) \leq f(y)\ \forall y \in S\} = O(\Upsilon_{\min}) \end{aligned}$$

hold. It is noted that (nd-sdp) $\Pi_{\min}$ strongly represents $\Upsilon_{\min}$ if it super-strongly represents $\Upsilon_{\min}$. Further, it weakly represents $\Upsilon_{\min}$, if it strongly represents $\Upsilon_{\min}$.

3. Super-strong Representation of an nd-ddp by an nd-sdp

Firstly, define some equivalence relations, which play an important role in super-strong representation theorems. For a given nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, $f(x) = \mathrm{Max}\{f_i(x) \mid i \in A(x)\}$, let us denote $\Sigma_A^* = \bigcup_{x \in \Sigma^*} \{(x,i) \mid i \in A(x)\} \subset \Sigma^* \times I^*$.

Definition 3.1 (equivalence relations)**.** For a given nd-ddp $\Upsilon_{\min}$, let us define the equivalence relations on $\Sigma^* \times I^*$ as follows:

$$\begin{aligned}
(x,i_x)\hat{R}_{S_{\bar{A}}}(y,i_y) &\iff \{(z,i_z) \mid (xz, i_x i_z) \in S_{\bar{A}}\} = \{(z,i_z) \mid (yz, i_y i_z) \in S_{\bar{A}}\}, \\
(x,i_x)\hat{R}_{f_i}(y,i_y) &\iff f_{i_x}(x) = f_{i_y}(y) \wedge (i_x \in \bar{A}(x), i_y \in \bar{A}(y)), \\
(x,i_x)\hat{R}_{\Upsilon_{f_i}}(y,i_y) &\iff (x,i_x)\hat{R}_{S_{\bar{A}}}(y,i_y) \wedge \ (\forall (xz, i_x i_z) \in S_{\bar{A}})(f_{i_x i_z}(xz) = f_{i_y i_z}(yz)).
\end{aligned}$$

An equivalence relation $\hat{R}$ on Σ_A^* is *right invariant* if $(x,i_x)\hat{R}(y,i_y) \implies (xz, i_x i_z)\hat{R}(yz, i_y i_z) \ \ \forall (z,i_z) \in \Sigma_A^*$. The equivalence relation $\hat{R}$ *refines* the set $S_{\bar{A}}$ if $(x,i_x)R(y,i_y) \implies ((x,i_x) \in S_{\bar{A}} \iff (y,i_y) \in S_{\bar{A}})$. Then $\Lambda(S_{\bar{A}})$ stands for all the right invariant equivalence relations which refine $S_{\bar{A}}$. In particular, $\Lambda(\Sigma_A^*)$ is the set of right invariant equivalence relations. We note that $\hat{R} = \hat{R}_{S_{\bar{A}}}, \ \hat{R}_{\Upsilon_{f_i}} \in \Lambda(S_{\bar{A}})$. Further define $\Lambda_F(S_{\bar{A}}) = \{\hat{T} \in \Lambda(S_{\bar{A}}) \mid |\Sigma_A^*/\hat{T}| < \infty\}$.

Then the following lemma will be used in deriving super-strong representation.

Lemma 3.1 (implementation of h' by nd-sdp)**.** For a given $x \in \Sigma^*$, let $A(x)$ be the set of sequences of index defined in $\Upsilon_{\min}$. For each x and each $i \in A(x), h'_i(x)$ be given. That is, $x \to h'_{A(x)}(x) = \{h'_i(x) \mid i \in A(x)\}$; set-valued function. For $A(x), h'_i(x)$, the equivalence relation $\hat{R}_{h'}$ on Σ_A^* is defined by

$$(x,i_x)\hat{R}_{h'}(y,i_y) \iff h'_{i_x}(x) = h'_{i_y}(y), i_x \in A(x), i_y \in A(y).$$

Then there exists an nd-sdp $\Pi_{\min} = (M, h, \xi_0, \min)$ satisfying that

$$\bar{h}_{q_0;\delta_x}(x) = h'_{i_x}(x), \ \ \forall (x,\delta_x) \in \Sigma_Y^*, \ \forall (x,i_x) \in \Sigma_A^*, \tag{3.1}$$

where, $\Sigma_Y^* = \bigcup_{x \in \Sigma^*} \{(x,\delta) \mid \delta \in Y(q_0,x)\}$, if and only if there exists $\hat{T} \in \Lambda_F(\Sigma_A^*)$ such that $\hat{T} \wedge \hat{R}_{h'} \in \Lambda(\Sigma_A^*)$.

Proof. Necessity. Let an nd-sdp $\Pi_{\min}$ satisfy the equation (3.1). Put $Q = I, \ Y(q_0,x) = A(x)$ for each $x \in \Sigma^*$ and define $\hat{T}$ on Σ_A^* by

$$(x,\delta_x)\hat{T}(y,\delta_y) \iff \pi(\delta_x) = \pi(\delta_y), \text{where } \delta_x \in Y(q_0,x), \ \delta_y \in Y(q_0,y). \tag{3.2}$$

Then we can show that $\hat{T} \in \Lambda_F(\Sigma_A^*)$. Further, define $\hat{R}_{\bar{h}_\delta}$, by

$$(x,i_x)\hat{R}_{\bar{h}_\delta}(y,i_y) \iff \bar{h}_{q_0;\delta_x}(x) = \bar{h}_{q_0;\delta_y}(y), \delta_x \in Y(q_0,x), \ \delta_y \in Y(q_0,y).$$

Then, we see that. for $\forall (a,r) \in \Sigma \times Q$

$$\begin{aligned}
&(x,\delta_x)(\hat{T} \wedge \hat{R}_{\bar{h}_\delta})(y,\delta_y) \\
&\implies (\pi(\delta_x) = \pi(\delta_y), \ \delta_x \in Y(q_0,x), \ \delta_y \in Y(q_0,y)) \ \wedge (\bar{h}_{q_0;\delta_x}(x) = \bar{h}_{q_0;\delta_y}(y)) \\
&\implies (\pi(\delta_{xa}) = \pi(\delta_x r) = \pi(\delta_y r) = \pi(\delta_{ya}) = r, \ \delta_x r \in Y(q_0,xa), \ \delta_y r \in Y(q_0,ya)) \\
&\quad \wedge(\bar{h}_{q_0;\delta_x r}(xa) = h(\bar{h}_{q_0;\delta_x}(x), \pi(\delta_x), r, a) = h(\bar{h}_{q_0;\delta_y}(y), \pi(\delta_y), r, a) = \bar{h}_{q_0;\delta_y r}(ya)) \\
&\iff (xa, \delta_x r)(\hat{T} \wedge \hat{R}_{\bar{h}_\delta})(ya, \delta_y r).
\end{aligned}$$

Hence, $\hat{T} \wedge \hat{R}_{\bar{h}_\delta} \in \Lambda(\Sigma_A^*)$, which implies that $\hat{T} \wedge \hat{R}_{h'} \in \Lambda(\Sigma_A^*)$.

Sufficiency. Let $\hat{T} \wedge \hat{R}_{h'} \in \Lambda(\Sigma_A^*)$, and let $M = (Q, \Sigma, q_0, ST, Q_F)$ be defined as follows: $Q = \{[(x, i_x)] \mid (x, i_x) \in C_i,\ i = 1, 2, \cdots, n\}$ and $\Sigma_A^*/\hat{T} = \{C_1, C_2, \cdots C_n\}$, where $[(x, i_x)]$ denotes the state corresponding to the equivalence class of $\Sigma_A^*/\hat{T}$ containing (x, i_x), and $q_0 = [(\epsilon, \mu)]$. Q_F is not explicitly specified. $\delta([(x, i_x)], a) = \{[(xa, i_x j)] \mid (x, i_x) \in C_i,\ (xa, i_x j) \in C_j\}$, $ST = \Big\{([(x, i_x)], \delta([(x, i_x)], a), a) \mid (x, i_x) \in C_i \in \Sigma_A^*/\hat{T}, a \in \Sigma\Big\}$.
Next, for $\xi \in R^1, q \in Q, r \in Q, a \in \Sigma$, define a function h as follows:

$$h(\xi, q, r, a) = \begin{cases} h'_{i_x j}(xa), & \text{if } \exists (x, i_x) \in \Sigma_A^* \text{ such that } \xi = h'_{i_x}(x), \\ & \quad q = [(x, i_x)] \in Q, r = [(xa, i_x j)] \in Q, \\ \text{any real number}, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then h is well-defined, since, if there exists some $(y, i_y) \in \Sigma_A^*$ such that $\xi = h'_{i_y}(y), q = [(y, i_y)] \in Q, r = [(ya, i_y j)] \in Q$, then we obtain

$$\begin{aligned} &\xi = h'_{i_x}(x) = h'_{i_y}(y), i_x \in A(x), i_y \in A(y), \\ &q = [(x, i_x)] = [(y, i_y)] \implies (x, i_x)\hat{T}(y, i_y), \\ &r = [(xa, i_x j)] = [(ya, i_y j)] \implies (xa, i_x j)\hat{T}(ya, i_y j), \end{aligned}$$

which implies that $(x, i_x)(\hat{T} \wedge \hat{R}_{h'})(y, i_y)$. From $\hat{T} \wedge \hat{R}_{h'} \in \Lambda(\Sigma_A^*)$, it follows that $(xa, i_x j)(\hat{T} \wedge \hat{R}_{h'})(ya, i_y j)$. Hence we have $h'_{i_x j}(xa) = h'_{i_y j}(ya)$.

Finally, put $\xi_0 = h'_\mu(\epsilon)$. Consequently, the resulting $\Pi_{\min} = (M, h, \xi_0, \min)$ satisfies the equation (3.1). □

From this lemma, we have the next super-strong representation theorem by nd-sdp.

Theorem 3.1 (super-strong representation of nd-sdp)**.** For a given nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, there exists an nd-sdp $\Pi_{\min} = (M, h, \xi_0, \min)$ which super-strongly represents $\Upsilon_{\min}$ if and only if there exists $\hat{T} \in \Lambda_F(S_{\bar{A}})$ satisfying that

$$(\forall (x, i_x), (y, i_y) \in S_{\bar{A}})((x, i_x)(\hat{T} \wedge \hat{R}_{f_i})(y, i_y) \implies (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y). \tag{3.4}$$

Proof. Necessity. Let an nd-sdp $\Pi_{\min}$ super-strongly represent nd-ddp $\Upsilon_{\min}$. Put $Q = I, Y(q_0, x) = A(x),$ for each $x \in \Sigma^*$ and $Q_F = A_F$, and define $\hat{T}$ on Σ_A^* by (3.2) in Lemma 3.1. Then $\hat{T} \in \Lambda_F(S_{\bar{A}})$, since $\hat{T} \in \Lambda_F(\Sigma_A^*)$ and it refines the set $S_{\bar{A}}$ because

$$\begin{aligned} &(x, \delta_x)\hat{T}(y, \delta_y) \wedge \left((x, \delta_x) \in S_{\bar{A}} = \bigcup_{x \in S} \{(x, i) \mid i \in \bar{A}(x)\} = \bigcup_{x \in F(\Pi_{\min})} \{(x, \delta) \mid \delta \in \bar{Y}(q_0, x)\} \right) \\ &\implies \pi(\delta_x) = \pi(\delta_y) \in Q_F = A_F \implies (y, \delta_y) \in S_{\bar{A}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} &((x, \delta_x), (y, \delta_y) \in S_{\bar{A}}) \wedge (x, \delta_x)\hat{T}(y, \delta_y) \wedge \big(f_{\delta_x}(x) = f_{\delta_y}(y),\ \delta_x \in \bar{A}(x), \delta_y \in \bar{A}(y)\big) \\ &\implies (x, \delta_x)\hat{T}(y, \delta_y) \wedge \big(\bar{h}_{q0;\delta_x}(x) = \bar{h}_{q0;\delta_y}(y)\big) \\ &\implies (x, \delta_x)\hat{T}(y, \delta_y) \wedge (\forall (z, \delta_z) \in \Sigma_A^*)(\bar{h}_{q0;\delta_x\delta_z}(xz) = \bar{h}_{q0;\delta_y\delta_z}(yz)) \\ &\implies (x, \delta_x)\hat{R}_{S_{\bar{A}}}(y, \delta_y) \wedge (\forall (xz, \delta_x\delta_z) \in S_{\bar{A}}) \big(f_{\delta_x\delta_z}(xz) = \bar{h}_{q0;\delta_y\delta_z}(yz) = f_{\delta_y\delta_z}(yz)\big) \\ &\implies (x, \delta_x)\hat{R}_{\Upsilon_{f_i}}(y, \delta_y). \end{aligned}$$

Sufficiency. Let $M = (Q, \Sigma, q_0, ST, Q_F)$ be defined by the same way as in Lemma 3.1, where $Q_F = \{[(x, i_x)] \mid (x, i_x) \in S_{\bar{A}} = \bigcup_{x\in S}\{(x, i) \mid i \in \bar{A}(x)\}\}$. Then we have $F(\Pi_{\min}) = F(M) = S, \bar{Y}(q_0, x) \equiv \bar{A}(x)$ for $\forall x \in S$, since

$$\begin{aligned} & x \in F(\Pi_{\min}) \wedge \delta_x \in \bar{Y}(q_0, x) \\ \iff & (\exists \delta_x \in Y(q_0, x) \text{ s.t. } \pi(\delta_x) \in Q_F) \wedge (\pi(\delta_x) \in Q_F) \\ \iff & (x \in S) \wedge (\pi(\delta_x) = \pi(i_x) \in A_F) \\ \iff & (x \in S) \wedge (i_x \in \bar{A}(x)). \end{aligned}$$

Further, define a function, $x \to h'_{A(x)}(x) = \{h'_i(x) \mid i \in A(x)\}$: the set-valued function as follows:

(1) $h'_{i_x}(x) = f_{i_x}(x)$ if $(x, i_x) \in S_{\bar{A}}$;

(2) $(x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) = h'_{i_y}(y)) \iff (x, i_x)\hat{T}(y, i_y) \wedge (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y)$,

which is possible since it follows from the condition (3.4) that

$$\begin{aligned} & (\forall (x, i_x), \forall (y, i_y) \in S_{\bar{A}})(h'_{i_x}(x) = h'_{i_y}(y) \wedge (x, i_x)\hat{T}(y, i_y)) \\ \Longrightarrow & (f_{i_x}(x) = f_{i_y}(y) \wedge (x, i_x)\hat{T}(y, i_y)) \Longrightarrow ((x, i_x)\hat{R}_{f_i}(y, i_y) \wedge (x, i_x)\hat{T}(y, i_y)) \\ \Longrightarrow & (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y). \end{aligned}$$

Next, define $\hat{R}_{h'}$ by the same way as in Lemma 3.1. Then, from the condition (2), we have

$$\hat{T} \wedge \hat{R}_{h'} = \hat{T} \wedge \hat{R}_{\Upsilon_{f_i}}. \tag{3.5}$$

Hence, from (3.5) and $\hat{T}, \hat{R}_{\Upsilon_{f_i}} \in \Lambda(\Sigma^*_A)$, it follows that $\hat{T} \wedge \hat{R}_{h'} \in \Lambda(\Sigma^*_A)$. From Lemma 3.1, there exists an nd-sdp $\Pi_{\min}$ such that $\bar{h}_{q_0;\delta_x}(x) = h'_{i_x}(x), \quad \forall (x, \delta_x) \in \Sigma^*_Y, \ \forall (x, i_x) \in \Sigma^*_A$. So, from the condition (1), it follows that

$$\bar{h}_{q_0;\delta_x}(x) = f_{i_x}(x) \ \forall (x, \delta_x) \in \bar{F}(\Pi_{\min}), \ \forall (x, i_x) \in S_{\bar{A}}(x),$$

that is, $\Pi_{\min}$ super strongly represents $\Upsilon_{\min}$. □

Example 3.1 (nondeterministic shortest path problem)**.** Let us consider an nondeterministic associative shortest path problem. Firstly, this problem can be formulated as a nondeterministic discrete decision process as follows (see Figure 1): nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, $\Sigma = \{1, 2, \ldots, N\} \ni j$: next move to node j , $S = \{x \in \Sigma^* | x = yN, \ y \in \Sigma^*\}, A(x = j_1 j_2 \cdots j_k) = \{i \mid i = i_0 i_1 i_2 \cdots i_k, \ i_0 = [1, -], i_1 = [j_1, l_1], i_2 = [j_2, l_2], \ldots, i_k = [j_k, l_k]\}$, where $l_1, l_2, \ldots, l_k$ denotes the scenarios, 1, 2, 3; for example, scenario 1: heavy traffic, scenario 2: ordinary traffic, scenario 3: light traffic, respectively, and each index $i = [j, l]$ means that one meets to a scenario i after taking policy j. According to the scenarios, 1, 2, 3, arc lengths, $t^1_{j_k j_l}, t^2_{j_k j_l}, t^3_{j_k j_l} \in A \subset R^1$ are associated with each arc (j_k, j_l), respectively. $A_F = \{[N, 3]\}$, $\bar{A}(x = j_1 j_2 \cdots j_k) = \{i \mid i = i_0 i_1 i_2 \cdots i_k, \ \pi(i) = i_k = [N, 3]\}$. For each $i_x = i_0 i_1 i_1 \cdots i_{k-1} i_k \in \bar{A}(x = j_1 j_2 \cdots j_k N)$, where $i_0 = [1, -], i_1 = [j_1, l_1], i_2 = [j_2, l_2], \ldots, i_{k-1} = [j_{k-1}, l_{k-1}], i_k = [N, 3]$, the value $f_{i_x}(x)$ is defined by $f_{i_x}(x = j_1 j_2 \cdots j_k) = t^{l_1}_{1 j_1} \circ t^{l_2}_{j_1 j_2} \circ \cdots \circ t^{l_{k-1}}_{j_{k-2} j_{k-1}} \circ t^3_{j_{k-1} N}$, where $\circ : A \times A \longrightarrow A$: binary operation satisfies the associative law. $f(x = j_1 j_2 \cdots j_k N) = \mathrm{Max}\{f_i(x) \mid i \in \bar{A}(x)\} = \mathrm{Max}\ f_{\bar{A}(x)}(x) \Longrightarrow$ minimize.

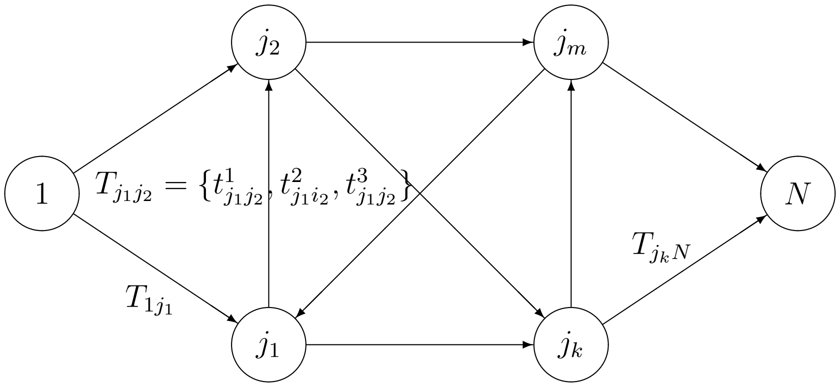

Figure 1: Nondeterministic shortest path problem (nd-ddp $\Upsilon_{\min}$)

For this problem, define an equivalent relation $\hat{T}$ by

$$(x, i_x)\hat{T}(y, i_y) \iff \begin{array}{l} x = j_1j_2\cdots j,\ y = j'_1j'_2\cdots j \\ i_x = i_0i_1i_2\cdots i,\ i_y = i_0i'_1i'_2\cdots i, \end{array}$$

then, $\hat{T} \in \Lambda_F(S_{\bar{A}})$. Further, since the assumption (3.4) in Theorem 3.1 is satisfied, $\Upsilon_{\min}$ is super-strongly represented by **nd-sdp** $\Pi_{min} = (M(Q, \Sigma, q_0, ST, Q_F), h, \xi_0, \min)$:

$$\begin{array}{l} Q = \{[1, -], [j_1, l_1], [j_2, l_2], \ldots, [j_k, l_k], \ldots [N, 3] \mid j_k : \text{node},\ l_k = 1 \text{ or } 2\text{or } 3\}, \\ q_0 = [1, -] :\ \text{initial node},\ \ Q_F = \{[N, 3]\}, \\ h(\xi, [j_k, l_k], [j_m, l_m], j_m) = \xi \circ t^{l_m}_{j_kj_m}, \quad t^{l_m}_{j_kj_m} \in T_{j_kj_m}, \\ \xi_0 = e(\circ)\text{:unit element of the binary operation } \circ. \end{array}$$

In fact, for $x = j_1j_2\ldots j_{k-1}N \in\ S$, and $r_m = [j_m, l_m],\ m = 1, 2, \cdots, k-1,\ r_k = [N, 3]$, it holds that

$$\begin{aligned} \bar{h}_{q_0;\mu r_1}(j_1) &= h(\xi_0, q_0, [j_1, l_1], j_1) = \xi_0 \circ t^{l_1}_{1j_1} = t^{l_1}_{1j_1}, \\ \bar{h}_{q_0;\mu r_1r_2}(j_1j_2) &= h(\bar{h}_{q_0;\mu r_1}(j_1), [j_1, l_1], [j_2, l_2], j_2) \\ &= \bar{h}_{q_0;\mu r_1}(j_1) \circ t^{l_2}_{j_1j_2} = t^{l_1}_{1j_1} \circ t^{l_2}_{j_1j_2}, \\ &\cdots \\ \bar{h}_{q_0;\mu r_1r_2\ldots r_{k-1}r_k}(j_1j_2\ldots j_{k-1}N) &= \bar{h}_{q_0;\mu r_1r_2\ldots r_{k-1}}(j_1j_2\ldots j_{k-1}) \circ t^3_{j_{k-1}N} \\ &= t^{l_1}_{1j_1} \circ t^{l_2}_{j_1j_2} \circ \cdots \circ t^{l_{k-1}}_{j_{k-2}j_{k-1}} \circ t^3_{j_{k-1}N} = f_i(x = j_1j_2\cdots j_{k-1}N), \end{aligned}$$

which implies that for $\forall(x, \delta_x) \in \bar{F}(\Pi_{\min}),\ \forall(x, i_x) \in S_{\bar{A}}$,

$$\bar{h}_{q_0;\delta_x}(x) = f_{i_x}(x),$$

that is, $\Pi_{\min}$ super-strongly represents $\Upsilon_{\min}$. It is noted that this Π_{min} is also nd-assdp.

4. Super-strong Representation of an nd-ddp by an nd-msdp and nd-pmsdp

The following lemma will be used in deriving super-strong representation theorem by nd-msdp and nd-pmsdp.

Lemma 4.1 (implementation of h' by nd-msdp)**.** Let $A(x)$, $h'_i(x)$ and $\hat{R}_{h'}$ be defined as in Lemma 3.1. Then there exists an nd-msdp $\Pi_{\min} = (M, h, \xi_0, \min)$ satisfying the equation (3.1), if and only if there exists $\hat{T} \in \Lambda_F(\Sigma_A^*)$ satisfying that

$$(x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) \leq h'_{i_y}(y)) \implies h'_{i_x i_z}(xz) \leq h'_{i_y i_z}(yz)\ (\forall (z, i_z) \in \Sigma_A^*). \tag{4.1}$$

Proof. Necessity. Let an nd-msdp $\Pi_{\min}$ satisfy the equation (3.1). Put $Q = I$, $Y(q_0, x) = A(x)\ \forall x \in \Sigma^*$ and define $\hat{T}$ by (3.2) in Lemma 3.1. Then $\hat{T} \in \Lambda_F(S_{\bar{A}})$, and it follows from the monotonicity of h and (3.1) that

$$\begin{aligned}
& (x, \delta_x)\hat{T}(y, \delta_y) \wedge h'_{\delta_x}(x) \leq h'_{\delta_y}(y) \\
\implies & \pi(\delta_x) = \pi(\delta_y), \text{where } \delta_x \in Y(q_0, x) = A(x),\ \delta_y \in Y(q_0, y) = A(y) \\
& \wedge\ \ \bar{h}_{q_0;\delta_x}(x) = h'_{\delta_x}(x) \leq h'_{\delta_y}(y) = \bar{h}_{q_0;\delta_y}(y) \\
\implies & \pi(\delta_x r_1) = \pi(\delta_y r_1), \text{where } \exists a_1 \in \Sigma \text{ s.t. } \delta_x r_1 \in Y(q_0, xa_1),\ \delta_y r_1 \in Y(q_0, ya_1) \\
& \wedge\ \ \bar{h}_{q_0;\delta_x r_1}(xa_1) = h(\bar{h}_{q_0;\delta_x}(x), \pi(\delta_x), r_1, a_1) \leq h(\bar{h}_{q_0;\delta_y}(y), \pi(\delta_y), r_1, a_1) = \bar{h}_{q_0;\delta_y r_1}(ya_1) \\
\implies & (xa_1, \delta_x r_1)\hat{T}(ya_1, \delta_y r_1)\ \wedge\ h'_{\delta_x r_1}(xa_1) \leq h'_{\delta_y r_1}(ya_1) \implies \cdots \\
\implies & h'_{\delta_x r_1 \cdots r_n}(xa_1 \cdots a_n) \leq h'_{\delta_y r_1 \cdots r_n}(ya_1 \cdots a_n).
\end{aligned}$$

Put $z = a_1 \cdots a_n \in \Sigma^*, \delta_z = r_1 \cdots r_n \in Y(\pi(\delta_x), z) = Y(\pi(\delta_y), z) \subset I^*$. Then we have (4.1).

Sufficiency. The condition (4.1) implies that $\hat{T} \wedge \hat{R}_{h'_a} \in \Lambda(\Sigma_A^*)$, where $\hat{R}_{h'_a}$ is defined by the same way as in Lemma 3.1, since

$$\begin{aligned}
& (x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) = h'_{i_y}(y)) \\
\iff & (x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) \leq h'_{i_y}(y)) \wedge (h'_{i_y}(y) \leq h'_{i_x}(x)) \\
\implies & (\forall (z, i_z) \in \Sigma_A^*)((xz, i_x i_z)\hat{T}(yz, i_y i_z) \wedge (h'_{i_x i_z}(xz) \leq h'_{i_y i_z}(yz)) \wedge (h'_{i_y i_z}(yz) \leq h'_{i_x i_z}(xz))) \\
\iff & (\forall (z, i_z) \in \Sigma_A^*)((xz, i_x i_z)\hat{T}(yz, i_y i_z) \wedge (h'_{i_x i_z}(xz) = h'_{i_y i_z}(yz))) \\
\implies & \hat{T} \wedge R_{h'_a} \in \Lambda(\Sigma_A^*).
\end{aligned}$$

Let $M = (Q, \Sigma, q_0, ST, Q_F)$ and h be defined in the same way as in the proof of the sufficiency of Lemma 3.1. If there exist $x, y \in \Sigma^*$ such that $q = [(x, i_x)] = [(y, i_y)]$ (i.e.$(x, i_x)\hat{T}(y, i_y)$), $r = [(xa, i_x j)] = [(ya, i_y j)]$ (i.e.$(xa, i_x j)\hat{T}(ya, i_y j)$) and $h'_{i_x}(x) = \xi_1 \leq h'_{i_y}(y) = \xi_2$, then we have

$$h(\xi_1, q, r, a) = h'_{i_x r}(xa) \leq h'_{i_y r}(ya) = h(\xi_2, q, r, a).$$

For the case that there exists no $x \in \Sigma^*$ such that $\xi = h'_{i_x}(x)(i_x \in A(x)), q = [(x, i_x)]$, $r = [(xa, i_x j)](i_x j \in A(xa))$, we can re-define the function h so that $h(\xi_1, q, r, a) \leq h(\xi, q, r, a) \leq h(\xi_2, q, r, a)$ holds for all ξ such that $\xi_1 = h'_{i_x}(x) \leq \xi \leq h'_{i_y}(y) = \xi_2$. Consequently, the resulting $\Pi_{\min}$ is an nd-msdp. □

Definition 4.1 (partial ordering relation)**.** For nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, A_F, \min)$, $f(x) = \mathrm{Max}\{f_i(x) \mid i \in \bar{A}(x)\}$, define a partial ordering relation $\preceq_{\Upsilon_{f_i}}$ on $S_{\bar{A}}$ as the following:

$$(x, i_x) \preceq_{\Upsilon_{f_i}} (y, i_y) \iff (x, i_x)\hat{R}_{S_{\bar{A}}}(y, i_y) \wedge\ (\forall (xz, i_x i_z) \in S_{\bar{A}})\ \ (f_{i_x i_z}(xz) \leq f_{i_y i_z}(yz)).$$

Proposition 4.1. The partial ordering relation $\preceq_{\Upsilon_{f_i}}$ is right invariant, and the following relation holds:

$$(x, i_x) \preceq_{\Upsilon_{f_i}} (y, i_y) \wedge (y, i_y) \preceq_{\Upsilon_{f_i}} (x, i_x) \iff (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y).$$

From Lemma 4.1, the following super-strong representation theorem for nd-msdp is derived.

Theorem 4.1 (super-strong representation of nd-msdp). For a given nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, there exists an nd-msdp $\Pi_{\min} = (M, h, \xi_0, \min)$ which super-strongly represents $\Upsilon_{\min}$ if and only if there exists $\hat{T} \in \Lambda_F(S_{\bar{A}})$ satisfying the following two conditions:

(i) $(\forall(x, i_x), (y, i_y) \in S_{\bar{A}})((x, i_x)(\hat{T} \wedge \hat{R}_{f_i})(y, i_y) \implies (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y))$;

(ii) $(x, i_x), (y, i_y) \in C_i \in \Sigma_A^*/\hat{T} \implies (x, i_x) \preceq_{\Upsilon_{f_i}} (y, i_y)$ or $(y, i_y) \preceq_{\Upsilon_{f_i}} (x, i_x)$.

Proof. Necessity. Let an nd-msdp $\Pi_{\min}$ super-strongly represent nd-ddp $\Upsilon_{\min}$. Put $Q = I, Y(q_0, x) = A(x)\ \forall x \in \Sigma^*$ and $Q_F = A_F$, and define $\hat{T}$ by the same way as in the proof of Lemma 3.1. Then $\hat{T} \in \Lambda_F(S_{\bar{A}})$ and satisfy the condition (i) by Theorem 3.1. Furthermore,

$$\begin{aligned}
&(x, \delta_x)\hat{T}(y, \delta_y) \wedge (\bar{h}_{q_0;\delta_x}(x) \leq \bar{h}_{q_0;\delta_y}(y)) \\
&\implies (x, \delta_x)\hat{T}(y, \delta_y) \wedge (\forall(z, \delta_z) \in \Sigma_A^*)(\bar{h}_{q_0;\delta_x\delta_z}(xz) \leq \bar{h}_{q_0;\delta_y\delta_z}(yz)) \\
&\implies (x, \delta_x)\hat{R}_{S_{\bar{A}}}(y, \delta_y) \wedge (\forall(xz, \delta_x\delta_z) \in S_{\bar{A}} = \bigcup_{x \in F(\Pi_{\min})} \{(x, \sigma) \mid \sigma \in \bar{Y}(q_0, x)\}) \\
&\quad (f_{\delta_x\delta_z}(xz) \leq f_{\delta_y\delta_z}(yz)) \iff (x, \delta_x) \preceq_{\Upsilon_{f_i}} (y, \delta_y).
\end{aligned}$$

Since $\bar{h}_{q_0;\delta_x}(x) \leq \bar{h}_{q_0;\delta_y}(y)$ or $\bar{h}_{q_0;\delta_y}(y) \leq \bar{h}_{q_0;\delta_x}(x)$ holds for each $(x, \delta_x), (y, \delta_y) \in \Sigma_A^*/\hat{T}$, so, we have the condition (ii).

Sufficiency. Let $M = (Q, \Sigma, q_0, ST, Q_F)$ be defined by the same way as in the proof of the sufficiency of Theorem 3.1. Then we have $F(\Pi_{\min}) = F(M) = S, \bar{Y}(q_0; x) \equiv \bar{A}(x)$ for $\forall x \in S$. Further, define the function $h'_{i_x}(x)$ on Σ_A^* as follows:

(1) $h'_{i_x}(x) = f_{i_x}(x)$ if $(x, i_x) \in S_{\bar{A}}$;

(2) $(x, i_x)\hat{T}(y, i_y) \wedge ((x, i_x) \preceq_{\Upsilon_{f_i}} (y, i_y)) \iff (x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) \leq h'_{i_y}(y))$,

which is possible since $\preceq_{\Upsilon_{f_i}}$ is a total ordering on each $C_i/\hat{R}_{\Upsilon_{f_i}}$ by Proposition 4.1 and condition (ii), where $C_i \in \Sigma_A^*/\hat{T}$, and

$$A_k \preceq_{\Upsilon_{f_i}} A_l \iff f_{i_x}(x) \leq f_{i_y}(y)$$

for $\forall(x, i_x) \in A_k, (y, i_y) \in A_l$, where $A_k,\ A_l \in C_i/\hat{R}_{\Upsilon_{f_i}}$, and $C_i \subset S_{\bar{A}}$; hence (1) does not contradict to (2).

Let $[(x, i_x)]$ represent the equivalent class in $\Sigma_A^*/\hat{R}_{\Upsilon_{f_i}}$, which contains $(x, i_x) \in \Sigma_A^*$. Then, from the condition (2), we obtain

$$\begin{aligned}
&(x, i_x)\hat{T}(y, i_y) \wedge h'_{i_x}(x) \leq h'_{i_y}(y) \\
&\iff (x, i_x)\hat{T}(y, i_y) \wedge [(x, i_x)] \preceq_{\Upsilon_{f_i}} [(y, i_y)] \\
&\iff (\forall(z, i_z) \in \Sigma_A^*)(xz, i_x i_z)\hat{T}(yz, i_y i_z) \wedge [(xz, i_x i_z)] \preceq_{\Upsilon_{f_i}} [(yz, i_y i_z)] \\
&\iff (\forall(z, i_z) \in \Sigma_A^*)(xz, i_x i_z)\hat{T}(yz, i_y i_z) \wedge h'_{i_x i_z}(xz) \leq h'_{i_y i_z}(yz).
\end{aligned}$$

Hence, by Lemma 4.1, there exists an nd-msdp $\Pi_{\min}$ such that $\bar{h}_{q_0;\delta_x}(x) = h'_{i_x}(x),\ \forall(x, \delta_x) \in \Sigma_Y^*,\ \forall(x, i_x) \in \Sigma_A^*$. So, it follows from the condition (1) that

$$\bar{h}_{q_0;\delta_x}(x) = f_{i_x}(x)\ \forall(x, \delta_x) \in \bar{F}(\Pi_{\min}),\ \forall(x, i_x) \in S_{\bar{A}}(x),$$

that is, nd-msdp $\Pi_{\min}$ super strongly represents $\Upsilon_{\min}$. □

In order to derive a super-strong representation by nd-pmsdp, let us introduce a directed graph $\hat{\Gamma}_{\gamma;\hat{T}}$ for nd-ddp and $\hat{T} \in \Lambda_F(S_{\bar{A}})$. Denote the set of equivalence classes of $\hat{R}_{\Upsilon_{f_i}} \wedge \hat{T}$ by $\hat{Y} = \Sigma_A^*/\hat{R}_{\Upsilon_{f_i}} \wedge \hat{T}$. Then, based on $\hat{Y}$, a directed graph $\hat{\Gamma}_{\gamma;\hat{T}}$ is defined as follows:

(1) $\hat{Y} \ni \hat{A}_i$: a node in $\hat{\Gamma}_{\Upsilon_m;T}$;

(2) $(\hat{A}_i, \hat{A}_j)$: an arc in $\hat{\Gamma}_{\gamma;\hat{T}}$ which has the following three types:

(a) arc of type A:

$\hat{A}_i \neq \hat{A}_j \wedge \hat{A}_i \hat{T} \hat{A}_j \wedge \hat{A}_i \preceq_{\Upsilon_{f_i}} \hat{A}_j$ or

$\hat{A}_i \neq \hat{A}_j \wedge (\hat{A}_i, \hat{A}_j \subset S_{\bar{A}}) \wedge f_{i_x}(x) < f_{i_y}(y)$ $(\forall (x, i_x) \in \hat{A}_i, \forall (y, i_y) \in \hat{A}_j)$;

(b) arc of type B:

$\hat{A}_i \neq \hat{A}_j \wedge (\hat{A}_i, \hat{A}_j \subset S_{\bar{A}}) \wedge f_{i_x}(x) < f_{i_y}(y)$ $(\forall (x, i_x) \in \hat{A}_i, \forall (y, i_y) \in \hat{A}_j)$;

(c) arc of type C: $\exists (a, i_a) \in \Sigma \times I$ s.t. $(xa, i_x i_a) \in \hat{A}_j$ $(\forall (x, i_x) \in \hat{A}_i)$.

A cycle in $\hat{\Gamma}_{\gamma;\hat{T}}$ is *inconsisitent* if it includes an arc of type A.

Theorem 4.2 (super-strong representation of nd-pmsdp). An nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$ is super-strongly representable by a nd-pmsdp $\Pi_{\min} = (M, h, \xi_0, \min)$ if and only if $\inf\{f(x) \mid x \in S\} > -\infty$ and there exists $\hat{T} \in \Lambda_F(S_{\bar{A}})$ satisfying the following three conditions:

(i) $(\forall (x, i_x), (y, i_y) \in S_{\bar{A}})((x, i_x)(\hat{T} \wedge \hat{R}_{f_i})(y, i_y) \implies (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y))$;

(ii) $(x, i_x), (y, i_y) \in C_i \in \Sigma_A^*/\hat{T} \implies$
$(x, i_x) \preceq_{\Upsilon_{f_i}} (y, i_y)$ or $(y, i_y) \preceq_{\Upsilon_{f_i}} (x, i_x)$;

(iii) graph $\hat{\Gamma}_{\gamma;\hat{T}}$ contains no inconsistent cycle.

Proof. Necessity. Let an nd-pmsdp $\Pi_{\min}$ super-strongly represent nd-ddp $\Upsilon_{\min}$. Put $Q = I, Y(q_0, x) = A(x)$ $\forall x \in \Sigma^*$ and $Q_F = A_F$, and define $\hat{T}$ by (3.2) in Lemma 3.1. Then $\hat{T} \in \Lambda_F(S_{\bar{A}})$ and satisfy the conditions (i), (ii) since it is nd-msdp by Theorem 4.1. Furthermore, from the condition, $h(\xi, q, r, a) \geq \xi$ for $\forall \xi, \forall (q, r, a) \in ST$, we have $\bar{h}_{q_0;\delta_x}(x) \geq \cdots \geq \bar{h}_{q_0;\mu r_1}(a_1) = h(\xi_{q_0}, \mu, r_1, a_1) \geq \xi_{q_0} = \bar{h}_{q_0;\mu}(\epsilon)$ for $\forall x$, $\forall \delta_x \in Y(q_0, x)$, which implies that $f(x) = \mathrm{Max}\, f_{A(x)}(x) = \mathrm{Max}\, \bar{h}_{q_0;Y(q_0,x)}(x) \geq \xi_{q_0} > -\infty \implies \inf\{f(x) \mid s \in S\} > -\infty$. Next, in order to prove the condition (iii), let

$$\beta = \{(\hat{A}_{i_1}, \hat{A}_{i_2}), (\hat{A}_{i_2}, \hat{A}_{i_3}), \ldots, (\hat{A}_{i_{k-1}}, \hat{A}_{i_k})\}, \ \hat{A}_{i_1} = \hat{A}_{i_k}$$

be an inconsistent cycle in the graph $\hat{\Gamma}_{\gamma;\hat{T}}$. Without loss of generality, we can assume that $(\hat{A}_{i_1}, \hat{A}_{i_2})$ is of type A. For a directed arc of type A, it holds that

$$(\forall (x, \delta_x) \in \hat{A}_i, \ \forall (y, \delta_y) \in \hat{A}_j) \left(\bar{h}_{q_0;\delta_x}(x) < \bar{h}_{q_0;\delta_y}(y)\right), \tag{4.2}$$

since, in case, $\hat{A}_i \neq \hat{A}_j \wedge \hat{A}_i \hat{T} \hat{A}_j \wedge \hat{A}_i \preceq_{\Upsilon_{f_i}} \hat{A}_j \implies \hat{A}_i \hat{T} \hat{A}_j \wedge (\sim \hat{A}_i \hat{R}_{\Upsilon_{f_i}} \hat{A}_j) \wedge \hat{A}_i \preceq_{\Upsilon_{f_i}} \hat{A}_j \implies$ (4.2), in case, $\hat{A}_i \neq \hat{A}_j \wedge (A_i, \hat{A}_j \in S_{\bar{A}}) \wedge (f_{i_x}(x) = \bar{h}_{q_0;\delta_x}(x) < f_{i_y}(y) = \bar{h}_{q_0;\delta_y}(y), \ \forall (x, i_x) \in \hat{A}_i, \ (y, i_y) \in \hat{A}_j) \implies$ (4.2). In the same way, we can show that for an arc $(\hat{A}_{i_1}, \hat{A}_{i_2})$ of type B, $(\forall (x, \delta_x) \in \hat{A}_i, \ \forall (y, \delta_y) \in \hat{A}_j) \left(\bar{h}_{q_0;\delta_x}(x) = \bar{h}_{q_0;\delta_y}(y)\right)$, and for an arc $(\hat{A}_{i_1}, \hat{A}_{i_2})$ of type C, $\forall (x, \delta_x), \exists (y, \delta_y) = (xa, \delta_x r)$ such that $\bar{h}_{q_0;\delta_x}(x) \leq \bar{h}_{q_0;\delta_x r}(xa) = \bar{h}_{q_0;\delta_y}(y)$. Consequently, for the inconsisitent cycle β, it is possible to select $(x^{i_j}, \delta_{x^{i_j}}) \in A_{i_j}, j = 1, 2, \cdots, k-1$ satisfying

$$\bar{h}_{q_0;\delta_{x^{i_1}}}(x^{i_1}) < \bar{h}_{q_0;\delta_{x^{i_2}}}(x^{i_2}) \leq \bar{h}_{q_0;\delta_{x^{i_3}}}(x^{i_3}) \leq \cdots \leq \bar{h}_{q_0;\delta_{x^{i_{k-1}}}}(x^{i_{k-1}}) \leq \bar{h}_{q_0;\delta_{x^{i_1}}}(x^{i_1}),$$

which is a contradiction.

Sufficiency. Let us define an equivalence relation $\doteq$ on $\hat{Y} = \Sigma_A^*/\hat{T}$ by

$$\hat{A}_i \doteq \hat{A}_j \iff (\exists \text{ paths both from } \hat{A}_i \text{ to } \hat{A}_j \text{ and from } \hat{A}_j \text{ to } \hat{A}_i \text{ in } \hat{\Gamma}_{\gamma_{f_i};\hat{T}}).$$

For equivalent classes $\hat{K}_p, \hat{K}_q \in \hat{Y}/\doteq$, define a partial ordering $\ll$ on $\hat{Y}/\doteq$ by

$$\hat{K}_p \ll \hat{K}_q \iff (\exists \text{ path from } \hat{A}_i \in \hat{K}_p \text{ to } \hat{A}_j \in \hat{K}_q \text{ in } \hat{\Gamma}_{\gamma_{f_i};\hat{T}}).$$

Let $\hat{W}' = \{\hat{K}_p \mid \hat{A}_i \setminus S_{\bar{A}} \neq \emptyset \text{ for } \forall \hat{A}_i \in \hat{K}_p\}$, $\hat{W} = \{(x, i_x) \in \Sigma_A^* \mid (x, i_x) \setminus S_{\bar{A}} \neq \emptyset\}$, where $(x, i_x) \setminus S_{\bar{A}} = \{(z, i_z) \in \Sigma_A^* \mid (xz, i_x i_z) \in S_{\bar{A}}\}$ and $\hat{A}_i \setminus S_{\bar{A}} = \{(z, i_z) \in \Sigma_A^* \mid \exists (x, i_x) \in \hat{A}_i \text{ s.t. } (xz, i_x i_z) \in S_{\bar{A}}\}$. It is noted that $\exists \hat{A}_i \in \hat{K}_p$ such that $\hat{A}_i \setminus S_{\bar{A}} \neq \emptyset \iff \hat{A}_i \setminus S_{\bar{A}} \neq \emptyset$ for $\forall \hat{A}_i \in \hat{K}_p$, and $(x, i_x) \in \hat{W} \iff \Big((x, i_x) \in \hat{A}_i, \hat{A}_i \in \hat{K}_p \Longrightarrow \hat{K}_p \in \hat{W}'\Big)$. Then consider a mapping $\gamma : \hat{W}'_p \to R^1$ such that

(a) $\gamma(\hat{K}_p) = f_{i_x}(x)\ \forall (x, i_x) \in C_i$ if $\exists \hat{A}_i \in \hat{K}_p$ s.t. $\hat{A}_i \subset S_{\bar{A}}$;
(b) $\hat{K}_p \ll \hat{K}_q \wedge \hat{K}_p \neq \hat{K}_q \implies \gamma(\hat{K}_p) < \gamma(\hat{K}_q)$;
(c) $\gamma(\hat{K}_0) < \gamma(\hat{K}_p)$ for all $\hat{K}_p \in \hat{Y}/\doteq$, where $\exists \hat{A}_0 \in \hat{K}_0$ s.t. $(\epsilon, \mu) \in \hat{A}_0$.

It can be shown in the same way as in Theorem 12.5 of Ibaraki [1] that the above γ exists. Next define the function $h'_{i_x}(x)$ on Σ_A^* by:

$$h'_{i_x}(x) = \begin{cases} \gamma(\hat{K}_p), & \text{if } \exists \hat{A}_i \in \hat{K}_p \text{ such that } (x, i_x) \in \hat{A}_i \text{ and } \hat{K}_p \in \hat{W}', \\ h'_{i_y}(y), & \text{otherwise, where } (x, i_x) = (yz, i_y i_z), \text{ and } (y, i_y) \text{ is the longest} \\ & \text{prefix of } (x, i_x) \text{ satisfying } (y, i_y) \in W'. \end{cases} \tag{4.3}$$

Then the function $h'_{i_x}(x)$ satisfies the following:
(1) $h'_{i_x}(x) = f_{i_x}(x)$ if $(x, i_x) \in S_{\bar{A}}$;
(2) $(x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) \leq h'_{i_y}(y)) \Longrightarrow h'_{i_x i_z}(xz) \leq h'_{i_y i_z}(yz) (\forall (z, i_z) \in \Sigma_A^*)$;
(3) $h'_{i_x}(x) \leq h'_{i_x i_y}(xy)$ for $\forall (x, i_x), \forall (y, i_y) \in \Sigma_A^*$.

From the properties, (a), (b), and (c), we can show (1) and (2) in the same way as in Theorem 12.5 of Ibaraki[2]. The last statement is proved as follows. Let $(x, i_x) \in \hat{A}_i \in \hat{K}_p$, $(xy, i_x i_y) \in \hat{A}_j \in \hat{K}_q$. Then, since there exists an directed arc of type C from $\hat{A}_i$ to $\hat{A}_j$, it holds that $\hat{K}_p \ll \hat{K}_q$. So, from the the property (b), $h'_{i_x}(x) = \gamma(\hat{K}_p) < \gamma(\hat{K}_q) = h'_{i_x i_y}(xy)$ if (x, i_x), $(xy, i_x i_y) \in \hat{W}$. In case $(x, i_x) \in \hat{W}, (xy, i_x i_y) \notin \hat{W}$, putting $(xy, i_x i_y) = (xzz', i_x i_z i'_z)$ where $(xz, i_x i_z) \in \hat{W}$, we have $h'_{i_x}(x) \leq h'_{i_x i_z}(xz) = h'_{i_x i_z i'_z}(xzz') = h'_{i_x i_y}(xy)$.

Hence, by (2) and Lemma 4.1, we obtain that there exists an nd-msdp $\Pi_{\min}$ such that $Y(q_0, x) \equiv A(x)$ and $\bar{h}_{q_0;Y(q_0;x)}(x) = h'_{A(x)}(x)$, $\forall x \in \Sigma^*$. So, from (1), it follows that

$$\bar{h}_{q_0;\bar{Y}(q_0;x)}(x) = h'_{\bar{A}(x)}(x) = f_{\bar{A}(x)}(x) \quad \forall x \in S.$$

Finally, from (3) it concludes that the nd-msdp $\Pi_{\min}$ is a nd-pmsdp, since

$$\xi = \bar{h}_{q_0;\delta_x}(x) \leq \bar{h}_{q_0;\delta_x r}(xa) = h(\xi, \pi(\delta_x), r, a).$$

□

Example 4.1 (egg dropping problem)**.** Suppose that we wish to know which windows in a k-story building are safe to drop eggs from, and which will cause the eggs to break on landing. Suppose m eggs are available. What is the least number of eggs-droppings that is guaranteed to work in all cases ? Let us assume that we can reuse an unbroken egg and can not use the broken eggs. Further, assume that we can not decide the minimum story and the least number of eggs-droppings if we can not find the minimum story although we have egg. First, this problem can be formulated by the following nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$; $\Sigma = \{1, 2, \ldots, k\} \ni j$: next drop an egg from j story, $\Sigma^* \ni x = 23$: sequence of stories from which we drop eggs, $S = \{x \in \Sigma^* | \exists i_x \in \bar{A}(x) \text{ after } x\}$, $A(x = j_1 j_2 \cdots j_n) = \{i_x \mid i_x = i_0 i_1 \cdots i_n,\ i_0 = ([m], \{1, 2, \ldots, k\}),\ i_1 = ([m_1], \{i, \cdots, l\}), \ldots, i_n = ([m_n], \{i, \cdots, l\})\}$, where $[m_i]$ denotes the number of unbroken eggs, $\{i, \cdots, l\}$ represents the set of unconfirmed stories. Further, $\bar{A}(x = j_1 j_2 \cdots j_n) = \{i_x \mid i_x = i_0 i_1 i_2 \cdots i_n,, i_n = ([m_n],\ \emptyset)\}$, $f_{i_x}(x = j_1 j_2 \cdots j_n) = n$, $i_x \in \bar{A}(x), x \in S$, $f(x) = \text{Max}\{f_{i_x}(x) \mid i_x \in A(x) = \bar{A}(x)\} = \text{Max } f_{\bar{A}(x)}(x)$, $= \infty$ if $A(x) \neq \bar{A}(x)$. In case of 2 eggs and , 3-story building, see the Figure 2.

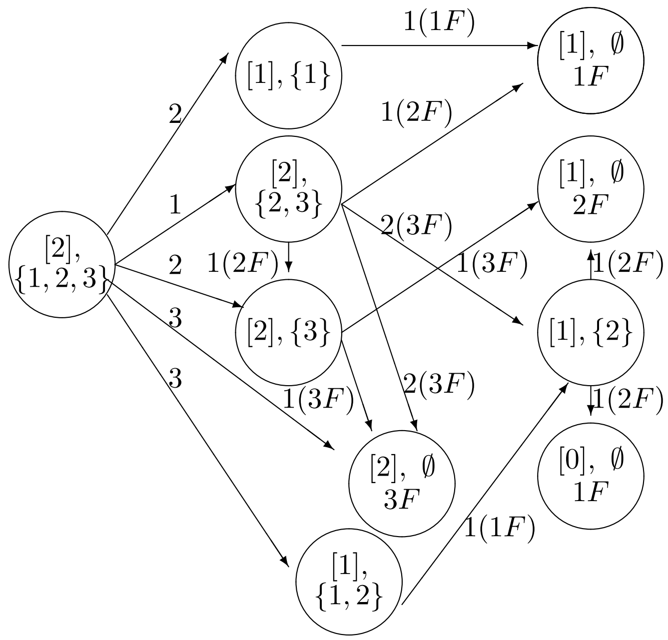

Figure 2: Egg-dropping problem formulated as nd-ddp

Note that $S = \{x = 3, 12, 21, 11, 111, 121, 311\}$ $f(3) = f(12) = f(11) = \infty$, $f(21) = 2$, $f(111) = f(121) = f(311) = 3$, $\min\{f(x)|x \in S\} = 2$. Further, define

$$(x, i_x)\hat{T}(y, i_y) \iff i_x = i_1 i_2 \cdots i \in A(x),\ i_y = j'_1 j'_2 \cdots i \in A(y)$$

Then, $\hat{T} \in \Lambda_F(S_{\bar{A}})$ and this equivalent relation $\hat{T}$ satisfies the condition of Theorem 4.2. Consequently, $\Upsilon_{\min}$ is super-strongly represented by the following positively nd-msdp (see Figure 3 and Figure 4):

nd-pmsdp $\Pi_{\min} = (M(Q, \Sigma, q_0, ST, Q_F), h, \xi_0, \min)$

$$Q = \{([m'], \{j_1, j_2, \ldots, j_n\})\}$$
$$q_0 = ([2], \{1, 2, 3\}),\quad Q_F = \{([m'],\ \emptyset)$$
$$h(\xi, q, r, j) = \xi + 1 > \xi \quad (\forall \xi),\quad \xi_0 = 0.$$

It holds that

$$\bar{h}_{q_0;\delta_x}(x) = f_{i_x}(x),\quad \forall (x, i_x) \in S_{\bar{A}}.$$

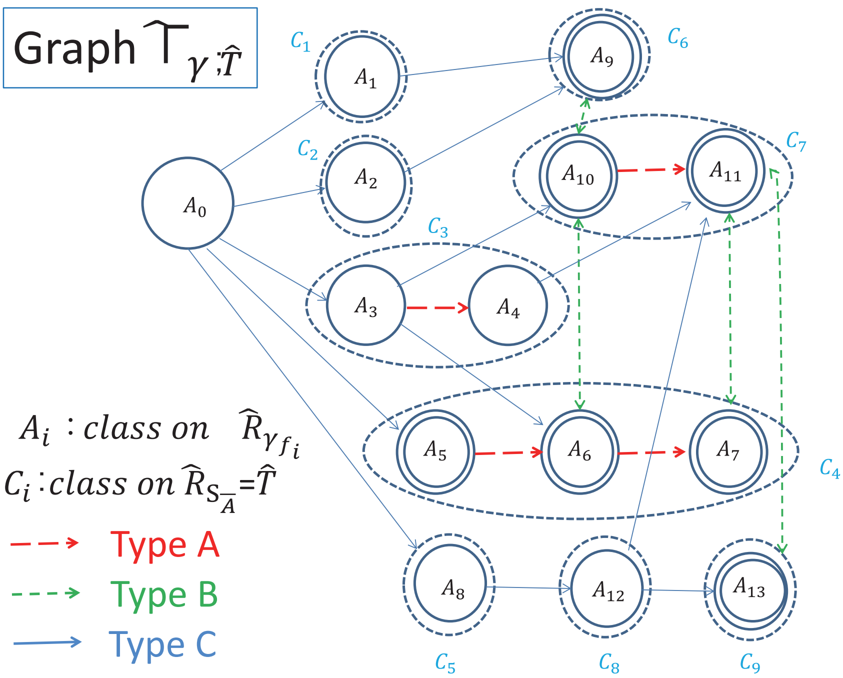

Figure 3: Graph $\hat{\Gamma}_{\gamma;\hat{T}}$

Here we note that $F(\Pi_{\min}) = \{x = 3, 12, 21, 11, 111, 121, 311\}$ $\bar{h}(3) = \bar{h}(12) = \bar{h}(11) = \infty$, $\bar{h}(21) = 2$, $\bar{h}(111) = \bar{h}(121) = \bar{h}(311) = 3$, $\min\{\bar{h}(x)|x \in F(\Pi_{\min})\} = 2$.

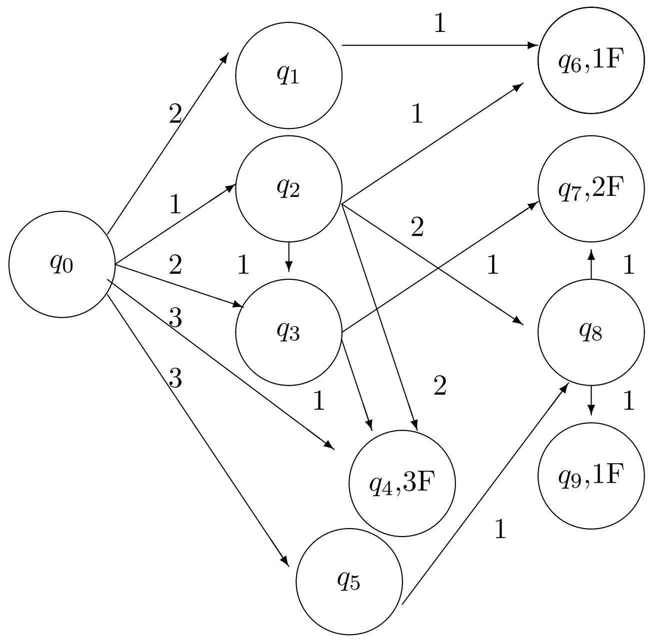

Figure 4: egg-dropping problem formulated as nd-pmsdp

5. Super-strong Representation of an nd-ddp by an nd-smsdp

The following lemma will be used in deriving super-strong representation theorem by nd-smsdp.

Lemma 5.1 (implementation of h' by nd-smsdp)**.** Let $A(x)$, $h'_i(x)$ and $\hat{R}_{h'}$ be defined as in Lemma 3.1. Then there exists an nd-smsdp $\Pi_{\min} = (M, h, \xi_0, \min)$ satisfying the equation (3.1), if and only if there exists $\hat{T} \in \Lambda_F(\Sigma_A^*)$ satisfying the following two conditions:

(i) $(x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) \leq h'_{i_y}(y)) \implies$

$$h'_{i_x i_z j}(xz) = h'_{i_y i_z}(yz) \ (\forall(z, i_z) \in \Sigma_A^*);$$

(ii) $(x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) < h'_{i_y}(y)) \implies$

$$h'_{i_x i_z j}(xz) < h'_{i_y i_z}(yz) \ (\forall(z, i_z) \in \Sigma_A^*).$$

Proof. Necessity. Let an nd-smsdp $\Pi_{\min}$ satisfy the equation (3.1). Put $Q = I, Y(q_0, x) = A(x) \ \forall x \in \Sigma^*$ and define $\hat{T}$ on Σ_A^* in the same way as in Lemma 3.1. Then we have $\hat{T} \in \Lambda_F(\Sigma_A^*)$, and it follows from (3.1) that

$$
\begin{aligned}
& \text{(i)} \;\; (x,\delta_x)\hat{T}(y,\delta_y) \wedge h'_{\delta_x}(x) = h'_{\delta_y}(y) \\
\Longrightarrow \;\; & \pi(\delta_x) = \pi(\delta_y), \text{where } \delta_x \in Y(q_0,x) = A(x),\;\; \delta_y \in Y(q_0,y) = A(y) \\
& \wedge \;\; \bar{h}_{q_0;\delta_x}(x) = h'_{\delta_x}(x) = h'_{\delta_y}(y) = \bar{h}_{q_0;\delta_y}(y) \\
\Longrightarrow \;\; & \pi(\delta_x r_1) = \pi(\delta_y r_1), \text{where } \exists a_1 \in \Sigma \text{ s.t. } \delta_x r_1 \in Y(q_0, xa_1),\;\; \delta_y r_1 \in Y(q_0, ya_1) \\
& \wedge \;\; \bar{h}_{q_0;\delta_x r_1}(xa_1) = h(\bar{h}_{q_0;\delta_x}(x), \pi(\delta_x), r_1, a_1) = h(\bar{h}_{q_0;\delta_y}(y), \pi(\delta_y), r_1, a_1) = \bar{h}_{q_0;\delta_y r_1}(ya_1) \\
\Longrightarrow \;\; & (xa_1, \delta_x r_1)\hat{T}(ya_1, \delta_y r_1) \;\wedge\; h'_{\delta_x r_1}(xa_1) = h'_{\delta_y r_1}(ya_1) \;\Longrightarrow\; \cdots \\
\Longrightarrow \;\; & h'_{\delta_x r_1 \cdots r_n}(xa_1 \cdots a_n) = h'_{\delta_y r_1 \cdots r_n}(ya_1 \cdots a_n).
\end{aligned}
$$

Further, from the strict monotonicity of h, we have

$$
\begin{aligned}
& \text{(ii)} \;\; (x,\delta_x)\hat{T}(y,\delta_y) \wedge h'_{\delta_x}(x) \le h'_{\delta_y}(y) \\
\Longrightarrow \;\; & \pi(\delta_x) = \pi(\delta_y), \text{where } \delta_x \in Y(q_0,x) = A(x),\;\; \delta_y \in Y(q_0,y) = A(y) \\
& \wedge \;\; \bar{h}_{q_0;\delta_x}(x) = h'_{\delta_x}(x) < h'_{\delta_y}(y) = \bar{h}_{q_0;\delta_y}(y) \\
\Longrightarrow \;\; & \pi(\delta_x r_1) = \pi(\delta_y r_1), \text{where } \delta_x r_1 \in Y(q_0, xa_1) = A(xa_1),\;\; \delta_y r_1 \in Y(q_0, ya_1) = A(ya_1) \\
& \wedge \;\; \bar{h}_{q_0;\delta_x r_1}(xa_1) = h(\bar{h}_{q_0;\delta_x}(x), \pi(\delta_x), r_1, a_1) < h(\bar{h}_{q_0;\delta_y}(y), \pi(\delta_y), r_1, a_1) = \bar{h}_{q_0;\delta_y r_1}(ya_1) \\
\Longrightarrow \;\; & (xa_1, \delta_x r_1)\hat{T}(ya_1, \delta_y r_1) \;\wedge\; h'_{\delta_x r_1}(xa_1) < h'_{\delta_y r_1}(ya_1) \;\Longrightarrow\; \cdots \\
\Longrightarrow \;\; & h'_{\delta_x r_1 \cdots r_n}(xa_1 \cdots a_n) < h'_{\delta_y r_1 \cdots r_n}(ya_1 \cdots a_n).
\end{aligned}
$$

Put $z = a_1 \cdots a_n \in \Sigma^*, \delta_z = r_1 \cdots r_n \in Y(\pi(\delta_x), z) = Y(\pi(\delta_y), z) \in I^*$, which implies that the conditions (i) and (ii).

Sufficiency. The condition (i) implies that $\hat{T} \wedge \hat{R}_{h'_a} \in \Lambda(\Sigma^*_A)$, which is defined in the same way as in Lemma 3.1, since (i) $\Longleftrightarrow \hat{T} \wedge \hat{R}_{h'_a} \in \Lambda(\Sigma^*_A)$.

Let $M = (Q, \Sigma, q_0, ST, Q_F)$ and the function h be defined in the same way as in the proof of the sufficiency of Lemma 3.1, then h is well-defined. If there exist $x, y \in \Sigma^*$ such that $q = [(x, i_x)] = [(y, i_y)]$ $((x, i_x)\hat{T}(y, i_y))$,
$r = [(xa, i_x j)] = [(ya, i_y j)]$ $((xa, i_x j)\hat{T}(ya, i_y j))$ and $h'_{i_x}(x) = \xi_1 < h'_{i_y}(y) = \xi_2$, then, by assumption (ii), we have

$$h(\xi_1, q, r, a) = h'_{i_x j}(xa) < h'_{i_y j}(ya) = h(\xi_2, q, r, a).$$

For the case that there exists no $x \in \Sigma^*$ such that $\xi = h'_{i_x}(x)(i_x \in A(x)), q = [(x, i_x)],\;\; r = [(xa, i_x j)](i_x j \in A(xa))$, we can re-define the function h so that $h(\xi_1, q, r, a) < h(\xi, q, r, a) < h(\xi_2, q, r, a)$ holds for all ξ such that $\xi_1 = h'_{i_x}(x) < \xi < h'_{i_y}(y) = \xi_2$. Consequently, the resulting $\Pi_{\min}$ is an nd-smsdp. □

Definition 5.1 (partial ordering relation)**.** For nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, $f(x) = \mathrm{Max}\{f_i(x) \mid i \in \bar{A}(x)\}$, define a partial ordering relation $\sqsubseteq_{\Upsilon_{f_i}}$ on $S_{\bar{A}}$ as follows:

$$
\begin{aligned}
(x, i_x) \sqsubseteq_{\Upsilon_{f_i}} (y, i_y) \quad \Longleftrightarrow \quad & (x, i_x)\hat{R}_{S_{\bar{A}}}(y, i_y) \wedge ((x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y)) \\
& \vee (\forall (xz, i_x i_z) \in S_{\bar{A}}) \;\; (f_{i_x i_z}(xz) < f_{i_y i_z}(yz)).
\end{aligned}
$$

Proposition 5.1. The partial ordering relation $\sqsubseteq_{\Upsilon_{f_i}}$ is right invariant, and the following relation holds:

$$(x, i_x) \sqsubseteq_{\Upsilon_{f_i}} (y, i_y) \wedge (y, i_y) \sqsubseteq_{\Upsilon_{f_i}} (x, i_x) \iff (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y).$$

From the Lemma 5.1, we obtain the following super-strong representation theorem for nd-smsdp:

Theorem 5.1 (super-strong representation of nd-smsdp). For a given nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, there exists an nd-smsdp $\Pi_{\min} = (M, h, \xi_0, \min)$ which super-strongly represents $\Upsilon_{\min}$ if and only if there exists $\hat{T} \in \Lambda_F(S_{\bar{A}})$ satisfying the following two conditions:

(i) $(\forall (x, i_x), (y, i_y) \in S_{\bar{A}})((x, i_x)(\hat{T} \wedge \hat{R}_{f_i})(y, i_y) \implies (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y))$;

(ii) $(x, i_x), (y, i_y) \in C_i \times I_i \in \Sigma_{A}^{*}/\hat{T} \implies (x, i_x) \sqsubseteq_{\Upsilon_{f_i}} (y, i_y)$ or $(y, i_y) \sqsubseteq_{\Upsilon_{f_i}} (x, i_x)$.

Proof. Necessity. Let an nd-smsdp $\Pi_{\min}$ super-strongly represent nd-ddp $\Upsilon_{\min}$ and $\hat{T}$ be defined by the same way as in Lemma 3.1. Then $\hat{T} \in \Lambda_F(S_{\bar{A}})$ and satisfy the condition (i) by Theorem 3.1. Furthermore,

$$
\begin{aligned}
&(x, \delta_x)\hat{T}(y, \delta_y) \wedge \bar{h}_{q0;\delta_x}(x) \leq \bar{h}_{q0;\delta_y}(y) \\
&\implies ((x, \delta_x)\hat{T}(y, \delta_y) \wedge (\bar{h}_{q0;\delta_x}(x) = \bar{h}_{q0;\delta_y}(y)) \vee ((x, \delta_x)\hat{T}(y, \delta_y) \wedge \bar{h}_{q0;\delta_x}(x) < \bar{h}_{q0;\delta_y}(y)) \\
&\implies \Big((x, \delta_x)\hat{R}_{S_{\bar{A}}}(y, \delta_y) \wedge (\forall (xz, \delta_x\delta_z) \in S_{\bar{A}}))(\bar{h}_{q0;\delta_x\delta_z}(xz) = \bar{h}_{q0;\delta_y\delta_z}(yz)\Big) \\
&\qquad \vee \Big((x, \delta_x)\hat{R}_{S_{\bar{A}}}(y, \delta_y) \wedge (\forall (xz, \delta_x\delta_z) \in S_{\bar{A}}))(\bar{h}_{q0;\delta_x\delta_z}(xz) < \bar{h}_{q0;\delta_y\delta_z}(yz)\Big) \\
&\iff (x, \delta_x)\hat{R}_{\Upsilon_{f_i}}(y, \delta_y) \vee \Big((x, \delta_x)\hat{R}_{S_{\bar{A}}}(y, \delta_y) \wedge (\forall (xz, \delta_x\delta_z) \in S_{\bar{A}})(f_{\delta_x\delta_z}(xz) < f_{\delta_y\delta_z}(yz))\Big) \\
&\iff \Big((x, \delta_x)\hat{R}_{\Upsilon_{f_i}}(y, \delta_y) \vee (x, \delta_x)\hat{R}_{S_{\bar{A}}}(y, \delta_y)\Big) \\
&\qquad \wedge \Big((x, \delta_x)\hat{R}_{\Upsilon_{f_i}}(y, \delta_y) \vee (\forall (xz, \delta_x\delta_z) \in S_{\bar{A}})(f_{\delta_x\delta_z}(xz) < f_{\delta_y\delta_z}(yz))\Big) \\
&\implies (xz, \delta_x\delta_z)\hat{R}_{S_{\bar{A}}}(yz, \delta_y\delta_z) \\
&\wedge \Big((x, \delta_x)\hat{R}_{\Upsilon_{f_i}}(y, \delta_y) \vee (\forall (xz, \delta_x\delta_z) \in S_{\bar{A}})(f_{\delta_x\delta_z}(xz) < f_{\delta_y\delta_z}(yz)\Big) \implies (x, \delta_x) \sqsubseteq_{\Upsilon_{f_i}} (y, \delta_y).
\end{aligned}
$$

Since $\bar{h}_{q0;\delta_x}(x) \leq \bar{h}_{q0;\delta_y}(y)$ or $\bar{h}_{q0;\delta_y}(y) \leq \bar{h}_{q0;\delta_x}(x)$ holds for each $(x, \delta_x), (y, \delta_y) \in \Sigma_{A}^{*}/\hat{T}$, so, we obtain the condition (ii).

Sufficiency. Let $M = (Q, \Sigma, q_0, ST, Q_F)$ be defined by the same way as in the proof of the sufficiency of Lemma 3.1. Then we have $F(\Pi_{\min}) = F(M) = S, \bar{Y}(q_0, x) \equiv \bar{A}(x)$ for $\forall x \in S$.

Further, define the function $h'_{i_x}(x)$ on Σ_{A}^{*} as follows:

(1) $h'_{i_x}(x) = f_{i_x}(x)$ if $(x, i_x) \in S_{\bar{A}}$;

(2) $(x, i_x)\hat{T}(y, i_y) \wedge (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y)) \iff (x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) = h'_{i_y}(y))$;

(3) $(x, i_x)\hat{T}(y, i_y) \wedge \Big((\sim (x, i_x)\hat{R}_{\Upsilon_{f_i}}(y, i_y)) \wedge (x, i_x) \sqsubseteq_{\Upsilon_{f_i}} (y, i_y)\Big)$

$$\iff (x, i_x)\hat{T}(y, i_y) \wedge (h'_{i_x}(x) < h'_{i_y}(y)),\ (i_x \in A(x), i_y \in A(y)),$$

which is possible since $\sqsubseteq_{\Upsilon_{f_i}}$ is a total ordering on each $C_i/\hat{R}_{\Upsilon_{f_i}}$ by Proposition 5.1 and condition (ii), where $C_i \in \Sigma_{A}^{*}/\hat{T}$, and

$$A_k \sqsubseteq_{\Upsilon_{f_i}} A_l \iff f_{i_x}(x) \leq f_{i_y}(y) \ \ \forall (x, i_x) \in A_k,\ (y, i_y) \in A_l$$

for $A_k,\ A_l \in C_i/\hat{R}_{\Upsilon_{f_i}}$, where $C_i \subset S_{\bar{A}}$; hence (1) does not contradict to (2) and (3).

From the condition (2), we obtain

$$\begin{aligned}
&(x,i_x)\hat{T}(x,i_x) \wedge h'_{i_x}(x) = h'_{i_y}(y)(i_x \in A(x), i_y \in A(y)) \\
\Longrightarrow\ & (x,i_x)\hat{T}(x,i_x) \wedge (x,i_x)\hat{R}_{\Upsilon_{f_i}}(y,i_y) \\
\Longrightarrow\ & (\forall (z,i_z) \in \Sigma_A^*)\Big((xz,i_xi_z)\hat{T}(yz,i_yi_z) \wedge (xz,i_xi_z)\hat{R}_{\Upsilon_{f_i}}(yz,i_yi_z)\Big) \\
\Longleftrightarrow\ & (\forall (z,i_z) \in \Sigma_A^*)\Big((xz,i_xi_z)\hat{T}(yz,i_yi_z) \wedge h'_{i_xi_z}(xz) = h'_{i_yi_z}(yz)\Big).
\end{aligned}$$

From (3) and Proposition 5.1, it follows that

$$\begin{aligned}
&(x,i_x)\hat{T}(x,i_x) \wedge h'_{i_x}(x) < h'_{i_y}(y)(i_x \in A(x), i_y \in A(y)) \\
\Longrightarrow\ & (x,i_x)\hat{T}(x,i_x) \wedge \Big((\sim (x,i_x)\hat{R}_{\Upsilon_{f_i}}(y,i_y)) \wedge (x,i_x) \sqsubseteq_{\Upsilon_{f_i}} (y,i_y)\Big) \\
\Longrightarrow\ & (\forall (z,i_z) \in \Sigma_A^*)\Big((xz,i_xi_z)\hat{T}(yz,i_yi_z) \\
&\qquad \wedge(\sim (xz,i_xi_z)\hat{R}_{\Upsilon_{f_i}}(yz,i_yi_z)) \wedge (xz,i_xi_z) \sqsubseteq_{\Upsilon_{f_i}} (yz,i_yi_z)\Big) \\
\Longleftrightarrow\ & (\forall (z,i_z) \in \Sigma_A^*)\Big((xz,i_xi_z)\hat{T}(yz,i_yi_z) \wedge h'_{i_xi_z}(xz) < h'_{i_yi_z}(yz)\Big).
\end{aligned}$$

Therefore, by Lemma 5.1, we obtain that there exists an nd-smsdp $\Pi_{\min}$ satisfying that (3.1). So, from the condition (1), it follows that

$$\bar{h}_{q0;\delta_x}(x) = f_{i_x}(x)\ \forall (x,\delta_x) \in \bar{F}(\Pi_{\min}),\ \forall (x,i_x) \in S_{\bar{A}}(x),$$

that is, nd-smsdp $\Pi_{\min}$ super strongly represents $\Upsilon_{\min}$. □

Definition 5.2 (equivalence relation $D^\circ_{\Upsilon_{f_i}}$). For nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$, $f(x) = \mathrm{Max}\{f_i(x) \mid i \in \bar{A}(x)\}$, let us define an equivalence relation $D^\circ_{\Upsilon_{f_i}}$ on $S_{\bar{A}}$ as the following:

$$\begin{aligned}
(x,i_x)D^\circ_{\Upsilon_{f_i}}(y,i_y) \iff\ & (x,i_x)\hat{R}_{S_{\bar{A}}}(y,i_y) \wedge (\forall (xw,i_xi_w),(xz,i_xi_z) \in S_{\bar{A}}) \\
& \big(f_{i_xi_w}(xw) \circ f_{i_yi_w}(yw)^{-1} = f_{i_xi_z}(xz) \circ f_{i_yi_z}(yz)^{-1}\big).
\end{aligned}$$

It is noted that if $\Upsilon_{\min}$ is an nd-ddp, then $D^\circ_{\Upsilon_{f_i}} \in \Lambda(S_{\bar{A}})$. By using this equivalence relation, we can derive the following super-strong representation theorem by nd-assdp:

Theorem 5.2 (super-strong representation of nd-assdp). A given nd-ddp $\Upsilon_{\min} = (\Sigma, S, f, \min)$ is super-strongly representable by an nd-assdp $\Pi_{\min} = (M, h, \xi_0, \min)$ if and only if it holds that

$$D^\circ_{\Upsilon_{f_i}} \in \Lambda_F(S_{\bar{A}}).$$

Proof. Necessity. Let an nd-assdp $\Pi_{\min}$ super-strongly represent nd-ddp $\Upsilon_{\min}$ and $\hat{T}$ be defined by (3.2) in Lemma 3.1. Then $\hat{T} \in \Lambda_F(S_{\bar{A}})$ and it follows that

$$\begin{aligned}
&(x,\delta_x)\hat{T}(y,\delta_y) \Longrightarrow (x,\delta_x)\hat{R}_{S_{\bar{A}}}(y,\delta_y) \wedge (\forall (xw,\delta_x\delta_w),\ \forall (xz,\delta_x\delta_z) \in S_{\bar{A}}) \\
&\quad (f_{\delta_x\delta_w}(xw) \circ f_{\delta_y\delta_w}(yw)^{-1} = \bar{h}_{q0;\delta_x\delta_w}(xw) \circ \bar{h}_{q0;\delta_y\delta_w}(yw)^{-1} \\
&\qquad = \bar{h}_{q0;\delta_x}(x) \circ \bar{h}_{q0;\delta_y}(y)^{-1} = \bar{h}_{q0;\delta_x\delta_z}(xz) \circ \bar{h}_{q0;\delta_y\delta_z}(yz)^{-1} = f_{\delta_x\delta_z}(xz) \circ f_{\delta_y\delta_z}(yz)^{-1}) \\
&\Longleftrightarrow\ (x,\delta_x)D^\circ_{\Upsilon_{f_i}}(y,\delta_y),
\end{aligned}$$

that is, it holds that $(x,\delta_x)\hat{T}(y,\delta_y) \implies (x,\delta_x)D^\circ_{\Upsilon_{f_i}}(y,\delta_y)$. Hence we have $|\Sigma_A^*/D^\circ_{\Upsilon_{f_i}}| \le |\Sigma_A^*/\hat{T}| < \infty$. Consequently, we see that $D^\circ_{\Upsilon_{f_i}} \in \Lambda_F(S_{\bar{A}})$.

Sufficiency. Put $\hat{T} = D^\circ_{\Upsilon_{f_i}}$ and define a function $h'_{i_x}(x)$ on Σ_A^* as the following:

(1) $h'_{i_x}(x) = f_{i_x}(x)$ if $(x,i_x) \in S_{\bar{A}}$;

(2) $(x,i_x)\hat{T}(y,i_y) \wedge ((x,i_x),\ (y,i_y) \in \hat{W}) \Longrightarrow (\forall(xw,i_xi_w) \in S_{\bar{A}})(h'_{i_x}(x) \circ h'_{i_y}(y)^{-1} = f_{i_xi_w}(xw) \circ f_{i_yi_w}(yw)^{-1})$, where $\hat{W} = \{(w,i_w) \in \Sigma_A^* \mid (w,i_w) \setminus S_{\bar{A}} \neq \emptyset\}$;

(3) $(x,i_x) \notin \hat{W} \Longrightarrow h'_{i_x}(x) = h'_{i_y}(y)$, where $(y,i_y) \in \Sigma_A^*$ is the longest policy such that $(x,i_x) = (yz,i_yi_z),\ (y,i_y) \in \hat{W}, (z,i_z) \in \Sigma_A^*$. Then, we can easily show that $\hat{T} \in \Lambda_F(\hat{W})$ and $h'_{i_x}(x)$ is well defined. Furthermore, for $h'_{i_x}(x)$ satisfy (1), (2) and (3), we have

$$
\begin{aligned}
&(x,i_x)\hat{T}(y,i_y) \wedge (h'_{i_x}(x) \le h'_{i_y}(y)),\ i_x \in A(x), i_y \in A(y) \Longrightarrow \\
&\quad (\forall(xz,i_xi_z), \forall(yz,i_yi_z) \in \hat{W}) \\
&\qquad (h'_{i_y}(y) \circ h'_{i_x}(x)^{-1} = f_{i_yi_zi_w}(yzw) \circ f_{i_xi_zi_w}(xzw)^{-1} = h'_{i_yi_z}(yz) \circ h'_{i_xi_z}(xz)^{-1})(\text{by}(2)) \\
&\quad (\forall(xz,i_xi_z), \forall(yz,i_yi_z) \notin \hat{W}) \\
&\qquad (h'_{i_yi_z}(yz) \circ h'_{i_xi_z}(xz)^{-1} = h'_{i_v}(v) \circ h'_{i_u}(u)^{-1} = h'_{i_y}(y) \circ h'_{i_x}(x)^{-1})(\text{by}(2),(3)),
\end{aligned}
$$

where $(u,i_u),(v,i_v)$ are the longest policy and indices such that $(xz,i_xi_z) = (uu',i_ui_{u'})$, $(yz,i_yi_z) = (vv',i_vi_{v'}), (u,i_u) \in \hat{W}, (v,i_v) \in \hat{W}, (u',i_{u'}), (v',i_{v'}) \in \Sigma_A^*$. Hence, if $(x,i_x)\hat{T}(y,i_y) \wedge (h'_{i_x}(x) \le h'_{i_y}(y)),\ i_x \in A(x), i_y \in A(y)$, then, for all $(z,i_z) \in \Sigma_A^*$,

$$
h'_{i_y}(y) \circ h'_{i_x}(x)^{-1} = h'_{i_yi_z}(yz) \circ h'_{i_xi_z}(xz)^{-1}. \tag{5.1}
$$

It follows from (5.1) that $h'_{i_yi_z}(yz) \circ h'_{i_y}(y)^{-1} = h'_{i_xi_z}(xz) \circ h'_{i_x}(x)^{-1}$, which implies that

$$
h'_{i_yi_z}(yz) - h'_{i_xi_z}(xz) = h'_{i_xi_z}(xz) \circ h'_{i_x}(x)^{-1} \circ h'_{i_y}(y)^{-1} - h'_{i_xi_z}(xz) \circ h'_{i_x}(x)^{-1} \circ h'_{i_x}(x) \ge 0,
$$

that is, $h'_{i_xi_z}(xz) \le h'_{i_yi_z}(yz)$ for all (z,i_z). Hence from Lemma 4.1, we have that there exists an nd-msdp $\Pi_{\min} = (M,h,\xi_0,\min)$ satisfying the equation (3.1). Finally, we will show that the nd-msdp $\Pi_{\min}$ is an nd-assdp; that is, $h(\xi,q,r,a) = \xi \circ \psi(q,r,a)$. For this purpose, let us prove that the value of $\psi(q,r,a) = h'_{i_xi_a}(xa) \circ h'_{i_x}(x)^{-1}$ is independent of x satisfying that $\pi(i_x) = q, \pi(i_xi_a) = r$. Let $(xa,i_xi_a),(ya,i_yi_a) \in \hat{W}, \pi(i_x) = \pi(i_y) = q, (xaw,i_xi_ai_w) \in S_{\bar{A}}$. Then from the condition (2) we have

$$
\begin{aligned}
h'_{i_xi_a}(xa) \circ h'_{i_x}(x)^{-1} &= h'_{i_yi_a}(ya) \circ f_{i_xi_ai_w}(xaw) \circ f_{i_yi_ai_w}(yaw)^{-1} \circ h'_{i_y}(y)^{-1} \\
&\quad \circ f_{i_yi_ai_w}(yaw) \circ f_{i_xi_ai_w}(xaw)^{-1} = h'_{i_yi_a}(ya) \circ h'_{i_y}(y)^{-1}.
\end{aligned} \tag{5.2}
$$

In the same way, we can prove that (5.2) holds for the case, $(xa,i_xi_a),(ya,i_yi_a) \notin \hat{W}$. □

Acknowledgements

The author would like to thank the referees for their several suggestions, very helpful comments and advices, which improve the first manuscript.

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