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# ON EQUIVALENCE OF M$^\natural$-CONCAVITY OF A SET FUNCTION AND SUBMODULARITY OF ITS CONJUGATE

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*Abstract* A fundamental theorem in discrete convex analysis states that a set function is M$^\natural$-concave if and only if its conjugate function is submodular. This paper gives an alternative direct proof to this fact.

**Keywords**: Combinatorial optimization, discrete convex analysis, M$^\natural$-concave function, valuated matroid, submodularity, conjugate function

## 1. Introduction

Let $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ be a set function on a finite set $N = \{1, 2, \ldots, n\}$, where the effective domain $\mathrm{dom}\, f = \{X \subseteq N \mid f(X) > -\infty\}$ is assumed to be nonempty. The conjugate function $g : \mathbb{R}^N \to \mathbb{R}$ of $f$ is defined by

$$g(p) = \max\{f(X) - p(X) \mid X \subseteq N\} \qquad (p \in \mathbb{R}^N), \tag{1.1}$$

where $p(X) = \sum_{i \in X} p_i$ (see Remark 1.1).

A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-concave* [13, 17] if, for any $X, Y \in \mathrm{dom}\, f$ and $i \in X \setminus Y$, it holds that*

$$f(X) + f(Y) \le f(X - i) + f(Y + i), \tag{1.2}$$

or there exists some $j \in Y \setminus X$ such that

$$f(X) + f(Y) \le f(X - i + j) + f(Y + i - j). \tag{1.3}$$

Since $f(X) + f(Y) > -\infty$ for $X, Y \in \mathrm{dom}\, f$, (1.2) requires $X - i, Y + i \in \mathrm{dom}\, f$, and (1.3) requires $X - i + j, Y + i - j \in \mathrm{dom}\, f$. A function $g : \mathbb{R}^N \to \mathbb{R}$ is called *submodular* if it satisfies the following inequality:

$$g(p) + g(q) \ge g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbb{R}^N), \tag{1.4}$$

where $p \vee q$ and $p \wedge q$ are the componentwise maximum and minimum of $p$ and $q$, respectively.

The following theorem states one of the most fundamental facts in discrete convex analysis [11, 13] that M$^\natural$-concavity of a set function $f$ can be characterized by submodularity of the conjugate function $g$.

**Theorem 1.** *A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is M$^\natural$-concave if and only if its conjugate function $g : \mathbb{R}^N \to \mathbb{R}$ is submodular.* ■

*We use short-hand notations such as $X - i = X \setminus \{i\}$, $Y + i = Y \cup \{i\}$, $X - i + j = (X \setminus \{i\}) \cup \{j\}$, and $Y + i - j = (Y \cup \{i\}) \setminus \{j\}$.

This theorem was first given by Danilov and Lang [3] in Russian; it is cited by Danilov, Koshevoy, and Lang [2]. It can also be derived through a combination of Theorem 10 of Ausubel and Milgrom [1] with the equivalence of gross substitutability and M$^\natural$-convexity due to Fujishige and Yang [8]. A self-contained detailed proof can be found in a recent survey paper by Shioura and Tamura [20, Theorem 7.2].

The objective of this paper is to give yet another proof to the above theorem. The proof does not use polyhedral-geometric characterizations of M$^\natural$-convex sets and functions, nor does it depend on the M-L conjugacy theorem in discrete convex analysis. Section 2 offers preliminaries from discrete convex analysis, and Section 3 presents the proof. Section 4 is a technical appendix.

**Remark 1.1.** The definition (1.1) of the conjugate function $g(p)$ here is consistent with its interpretation in economics. If $f(X)$ denotes the utility (or valuation) function for a bundle $X$, then $g(p)$ in (1.1) is the indirect utility function under the price vector $p$. In convex analysis, however, the conjugate of a concave function $f$ is more often defined as $f^*(p) = \min_x\{p^\top x - f(x)\}$. ■

## 2. Preliminaries on M-concave Functions

A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *valuated matroid* [4, 6] if, for any $X, Y \in \mathrm{dom}\, f$ and $i \in X \setminus Y$, there exists some $j \in Y \setminus X$ such that

$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j). \tag{2.1}$$

This property is referred to as the *exchange property*. A valuated matroid is also called an *M-concave set function* [9, 13]. The effective domain $\mathcal{B}$ of an M-concave function forms the family of bases of a matroid, and in particular, $\mathcal{B}$ consists of equi-cardinal subsets, i.e., $|X| = |Y|$ for all $X, Y \in \mathcal{B}$.

As is obvious from the definitions, M-concave functions form a subclass of M$^\natural$-concave functions.

**Proposition 2.** *A set function $f$ is M-concave if and only if it is an M$^\natural$-concave function and $|X| = |Y|$ for all $X, Y \in \mathrm{dom}\, f$.* ■

The concepts of M-concave and M$^\natural$-concave functions are in fact equivalent. For a function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$, we associate a function $\tilde{f}$ with an equi-cardinal effective domain. Denote by $r$ and $r'$ the maximum and minimum, respectively, of $|X|$ for $X \in \mathrm{dom}\, f$. Let $s \geq r - r'$, $S = \{n+1, n+2, \ldots, n+s\}$, and $\tilde{N} = N \cup S = \{1, 2, \ldots, \tilde{n}\}$, where $\tilde{n} = n + s$. We define $\tilde{f} : 2^{\tilde{N}} \to \mathbb{R} \cup \{-\infty\}$ by

$$\tilde{f}(Z) = \begin{cases} f(Z \cap N) & (|Z| = r), \\ -\infty & (\text{otherwise}). \end{cases} \tag{2.2}$$

Then, for $X \subseteq N$ and $U \subseteq S$, we have $\tilde{f}(X \cup U) = f(X)$ if $|U| = r - |X|$.

**Proposition 3.** *A set function $f$ is M$^\natural$-concave if and only if $\tilde{f}$ is M-concave.*

*Proof.* This fact is well known among experts. Since $f$ is a projection of $\tilde{f}$, the "if" part follows from [13, Theorem 6.15 (2)]. A proof of the "only-if" part can be found, e.g., in [16]. □

The exchange property for M-concave set functions is in fact equivalent to a local exchange property under some assumption on the effective domain. We say that a family $\mathcal{B}$ of equi-cardinal subsets is *connected* if, for any distinct $X, Y \in \mathcal{B}$, there exist $i \in X \setminus Y$ and

$j \in Y \setminus X$ such that $Y + i - j \in \mathcal{B}$. As is easily seen, $\mathcal{B}$ is connected if and only if, for any distinct $X, Y \in \mathcal{B}$ there exist distinct $i_1, i_2, \ldots, i_m \in X \setminus Y$ and $j_1, j_2, \ldots, j_m \in Y \setminus X$, where $m = |X \setminus Y| = |Y \setminus X|$, such that $Y \cup \{i_1, i_2, \ldots, i_k\} \setminus \{j_1, j_2, \ldots, j_k\} \in \mathcal{B}$ for $k = 1, 2, \ldots, m$.

The following theorem is a strengthening by Shioura [19, Theorem 2] of the local exchange theorem of Dress–Wenzel [5] and Murota [10] (see also [12, Theorem 5.2.25], [13, Theorem 6.4])$^\dagger$.

**Theorem 4.** *A set function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is M-concave if and only if*

(i) $\mathrm{dom}\, f$ *is a connected nonempty family of equi-cardinal sets, and*

(ii) *for any $X, Y \in \mathrm{dom}\, f$ with $|X \setminus Y| = 2$, there exist some $i \in X \setminus Y$ and $j \in Y \setminus X$ for which* (2.1) *holds.*

*Proof.* The "only-if" part is obvious. For the "if" part, the proof of Theorem 5.2.25 in [12, pp.295–297] works with the only modification in the proof of Claim 2 there. Since the proof is omitted in [19], we include the proof in Section 4. □

## 3. A Proof of Theorem 1

We prove the characterization of M$^\natural$-concavity by submodularity of the conjugate function (Theorem 1). Let $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ be a set function with $\mathrm{dom}\, f \neq \emptyset$, and $g : \mathbb{R}^N \to \mathbb{R}$ be its conjugate function, which is defined as $g(p) = \max\{f(X) - p(X) \mid X \subseteq N\}$ in (1.1).

We first show that M$^\natural$-concavity of $f$ implies submodularity of $g$.

**Lemma 5.** *If $f$ is M$^\natural$-concave, then $g$ is submodular.*

*Proof.* As is well known, $g$ is submodular if and only if

$$g(p + a\chi_i) + g(p + b\chi_j) \geq g(p) + g(p + a\chi_i + b\chi_j) \tag{3.1}$$

for any $p \in \mathbb{R}^N$, distinct $i, j \in N$, and $a, b \geq 0$, where $\chi_i$ and $\chi_j$ are the $i$th and $j$th unit vectors, respectively. For simplicity of notation we assume $p = \mathbf{0}$, and write $p^i = a\chi_i$, $p^j = b\chi_j$, and $p^{ij} = a\chi_i + b\chi_j$. Take $X, Y \subseteq N$ such that

$$g(p) = f(X) - p(X) = f(X), \quad g(p^{ij}) = f(Y) - p^{ij}(Y) = f(Y) - a|Y \cap \{i\}| - b|Y \cap \{j\}|.$$

Note also that $g(p^i) = \max\ \{f(Z) - a|Z \cap \{i\}| \mid Z \subseteq N\}$ and similarly for $g(p^j)$.

- If $|Y \cap \{i, j\}| = 2$, then $g(p) + g(p^{ij}) = (f(X) - a) + (f(Y) - b) \leq (f(X) - a|X \cap \{i\}|) + (f(Y) - b|Y \cap \{j\}|) \leq g(p^i) + g(p^j)$.
- If $|Y \cap \{i, j\}| = 1$, we may assume $i \in Y$ and $j \notin Y$. Then $g(p) + g(p^{ij}) = f(X) + (f(Y) - a) \leq (f(X) - a|X \cap \{i\}|) + (f(Y) - b|Y \cap \{j\}|) \leq g(p^i) + g(p^j)$.
- If $|Y \cap \{i, j\}| = 0$, then $g(p) + g(p^{ij}) = f(X) + f(Y)$. If $i \notin X$, we have $f(X) + f(Y) = (f(X) - a|X \cap \{i\}|) + (f(Y) - b|Y \cap \{j\}|) \leq g(p^i) + g(p^j)$. Similarly, if $j \notin X$. Suppose $\{i, j\} \subseteq X$. By the M$^\natural$-concave exchange property, we have $f(X) + f(Y) \leq f(X') + f(Y')$, where $(X', Y') = (X - i, Y + i)$ or $(X', Y') = (X - i + k, Y + i - k)$ for some $k \in Y \setminus X$. Since $i \notin X'$ and $j \notin Y'$, we have $f(X') + f(Y') = (f(X') - a|X' \cap \{i\}|) + (f(Y') - b|Y' \cap \{j\}|) \leq g(p^i) + g(p^j)$. □

Next, we show, in two steps, that submodularity of $g$ implies M$^\natural$-concavity of $f$. We treat the M-concave case in Lemmas 6 to 8, and the M$^\natural$-concave case in Lemma 9. It is emphasized that the combinatorial essence is captured in Lemma 7 for the M-concave case.

$^\dagger$In [5, 10], the effective domain is assumed to be a matroid basis family, and the assumption is weakened to connectedness in [19]. It is well known that a matroid basis family is connected.

**Lemma 6.** *If* dom $f$ *is a family of equi-cardinal sets and* $g$ *is submodular, then* dom $f$ *is connected.*

*Proof.* To prove this by contradiction, suppose that dom $f$ is not connected. Then there exist $X, Y \in \mathrm{dom}\, f$ such that $|X \setminus Y| = |Y \setminus X| \geq 2$ and there exists no $Z \in \mathrm{dom}\, f \setminus \{X, Y\}$ satisfying $X \cap Y \subseteq Z \subseteq X \cup Y$. Let $i_0$ be any element of $X \setminus Y$ and $j_0$ be any element of $Y \setminus X$. Let $M$ be a sufficiently large positive number in the sense that $M \gg n$ and $M \gg F$ for $F = \max\{|f(W)| \mid W \in \mathrm{dom}\, f\}$. Define $p, q \in \mathbb{R}^N$ by

$$p_i = \begin{cases} -M & (i = i_0), \\ 0 & (i \in (X \setminus Y) \setminus \{i_0\}), \end{cases} \qquad q_i = \begin{cases} 0 & (i = i_0), \\ -M & (i \in (X \setminus Y) \setminus \{i_0\}); \end{cases}$$

$$p_i = q_i = \begin{cases} -M & (i = j_0), \\ 0 & (i \in (Y \setminus X) \setminus \{j_0\}), \\ -M^2 & (i \in X \cap Y), \\ +M^2 & (i \in N \setminus (X \cup Y)). \end{cases}$$

Denote $m = |X \setminus Y|$ and $C = M^2|X \cap Y|$. Since there is no $Z \in \mathrm{dom}\, f \setminus \{X, Y\}$ satisfying $X \cap Y \subseteq Z \subseteq X \cup Y$, we have

$$\begin{aligned} g(p) &= \max\{f(X) - p(X), f(Y) - p(Y)\} \\ &= \max\{f(X) + M, f(Y) + M\} + C \\ &\leq F + M + C, \\ g(q) &= \max\{f(X) - q(X), f(Y) - q(Y)\} \\ &= \max\{f(X) + (m-1)M, f(Y) + M\} + C \\ &\leq F + (m-1)M + C, \end{aligned}$$

and therefore

$$g(p) + g(q) \leq 2F + mM + 2C. \tag{3.2}$$

Similarly, we have

$$\begin{aligned} g(p \vee q) &= \max\{f(X) - (p \vee q)(X), f(Y) - (p \vee q)(Y)\} \\ &= \max\{f(X), f(Y) + M\} + C \\ &= f(Y) + M + C, \\ g(p \wedge q) &= \max\{f(X) - (p \wedge q)(X), f(Y) - (p \wedge q)(Y)\} \\ &= \max\{f(X) + mM, f(Y) + M\} + C \\ &= f(X) + mM + C, \end{aligned}$$

and therefore

$$g(p \vee q) + g(p \wedge q) = f(X) + f(Y) + (m+1)M + 2C. \tag{3.3}$$

Since $M \gg F$, it follows from (3.2) and (3.3) that $g(p) + g(q) < g(p \vee q) + g(p \wedge q)$, which contradicts the submodularity of $g$. □

**Lemma 7.** *If* dom $f$ *is a family of equi-cardinal sets and* $g$ *is submodular, then* $f$ *has the local exchange property* (ii) *in Theorem 4.*

*Proof.* To prove by contradiction, suppose that the local exchange property fails for $X, Y$ with $|X \setminus Y| = |Y \setminus X| = 2$. To simplify notations we assume $X \setminus Y = \{1,2\}$ and $Y \setminus X = \{3,4\}$, and write $\alpha_{ij} = f((X \cap Y)+i+j)$, etc. Then we have $\alpha_{12}+\alpha_{34} > \max\{\alpha_{13}+\alpha_{24}, \alpha_{14}+\alpha_{23}\}$ by the failure of the local exchange property. Consider an undirected graph $G = (V,E)$ on vertex set $V = \{1,2,3,4\}$ and edge set $E = \{(i,j) \mid \alpha_{ij} > -\infty\}$. The graph $G$ has a unique maximum weight perfect matching $M = \{(1,2),(3,4)\}$ with respect the edge weight $\alpha_{ij}$. By duality (see Remark 3.1 below) there exists $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) \in \mathbb{R}^4$ such that $\alpha_{12} = \hat{p}_1 + \hat{p}_2$, $\alpha_{34} = \hat{p}_3 + \hat{p}_4$, and $\alpha_{ij} < \hat{p}_i + \hat{p}_j$ if $(i,j) \neq (1,2),(3,4)$. Define $\beta_{ij} = \alpha_{ij} - \hat{p}_i - \hat{p}_j$, to obtain $\beta_{12} = \beta_{34} = 0$ and $\beta_{ij} < 0$ if $(i,j) \neq (1,2),(3,4)$.

To focus on $\{1,2,3,4\}$ we partition $p$ into two parts as $p = (p', p'')$ with $p' \in \mathbb{R}^{\{1,2,3,4\}}$ and $p'' \in \mathbb{R}^{N \setminus \{1,2,3,4\}}$. We express $p' = \hat{p} + q$ with $q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$, while fixing $p''$ to the vector $\bar{p}$ defined by

$$\bar{p}_i = \begin{cases} -M & (i \in X \cap Y), \\ +M & (i \in N \setminus (X \cup Y)) \end{cases}$$

with a sufficiently large positive number $M$. Let $h(q) = g(\hat{p}+q, \bar{p}) - M|X \cap Y|$. By the choice of $\hat{p}$ and $\bar{p}$ as well as the assumed equi-cardinality of $\mathrm{dom}\, f$, we have

$$h(q) = \max\{\beta_{ij} - q_i - q_j \mid i,j \in \{1,2,3,4\},\ i \neq j\}$$

if $\|q\|_\infty$ is small enough compared with $M$. Let $a > 0$ be a (small) positive number with $a \leq \min\{|\beta_{ij}| \mid (i,j) \neq (1,2),(3,4)\}$. Then $h(0,0,0,0) = h(a,-a,0,0) = h(a,0,0,0) = 0$ and $h(0,-a,0,0) = a$. This shows a violation of submodularity of $h$, and hence that of $g$. □

Lemmas 6 and 7 with Theorem 4 show the following.

**Lemma 8.** *If* $\mathrm{dom}\, f$ *is a nonempty family of equi-cardinal sets and* $g$ *is submodular, then* $f$ *is an M-concave function.* ■

**Remark 3.1.** In general, the perfect matching polytope of a graph $G = (V,E)$ is described by the following system of equalities for $x \in \mathbb{R}^E$: (i) $x_e \geq 0$ for each $e \in E$, (ii) $x(\delta(v)) = 1$ for each $v \in V$, (iii) $x(\delta(U)) \geq 1$ for each $U \subseteq V$ with $|U|$ being odd $\geq 3$, where $\delta(v)$ denotes the set of edges incident to a vertex $v$ and $\delta(U)$ the set of edges between $U$ and $V \setminus U$; see Schrijver [18, Section 25.1]. In the proof of Lemma 7 we have $V = \{1,2,3,4\}$, in which case the inequalities of type (iii) are not needed, since $\delta(U) = \delta(v)$ for $U$ with $|U| = 3$ and the vertex $v \in V \setminus U$. Consider the maximum weight perfect matching problem on our $G = (V,E)$. This problem can be formulated in a linear program to maximize $\sum_{(i,j)\in E} \alpha_{ij} x_{ij}$ subject to $\sum_j x_{ij} = 1$ for $i = 1,2,3,4$ and $x_{ij} \geq 0$ for $(i,j) \in E$. Our assumption $\alpha_{12} + \alpha_{34} > \max\{\alpha_{13} + \alpha_{24}, \alpha_{14} + \alpha_{23}\}$ means that this problem has a unique optimal solution $x$ with $x_{12} = x_{34} = 1$ and $x_{ij} = 0$ for $(i,j) \neq (1,2),(3,4)$. The dual problem is to minimize $p_1 + p_2 + p_3 + p_4$ subject to $p_i + p_j \geq \alpha_{ij}$ for $(i,j) \in E$. The strict complementary slackness guarantees the existence of a pair of optimal solutions $(x_{ij} \mid (i,j) \in E)$ and $(p_i \mid i = 1,2,3,4)$ with the property that either $x_{ij} > 0$ or $p_i + p_j > \alpha_{ij}$ (exactly one of these) holds for each $(i,j) \in E$. Therefore, there exists $(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)$ such that $\alpha_{12} = \hat{p}_1 + \hat{p}_2$, $\alpha_{34} = \hat{p}_3 + \hat{p}_4$, and $\alpha_{ij} < \hat{p}_i + \hat{p}_j$ for $(i,j) \neq (1,2),(3,4)$. ■

Next we turn to the M$^\natural$-concave case. Consider the function $\tilde{f} : 2^{\tilde{N}} \to \mathbb{R} \cup \{-\infty\}$ of (2.2) associated with $f : 2^N \to \mathbb{R} \cup \{-\infty\}$, where $\tilde{N} = N \cup S$ and $\mathrm{dom}\, \tilde{f} \subseteq \{X \mid |X| = r\}$. We take $S$ with $|S| \geq r - r' + 2$. Let $\tilde{g}(p,q)$ denote the conjugate of $\tilde{f}$, where $p \in \mathbb{R}^N$ and $q \in \mathbb{R}^S$.

**Lemma 9.** *If $g$ is submodular, then $\tilde{g}$ is submodular.*

*Proof.* By definition,

$$\tilde{g}(p,q) = \max\{f(X) - p(X) - q(U) \mid X \subseteq N,\ U \subseteq S,\ |X| + |U| = r\}. \tag{3.4}$$

It suffices to prove that

$$\tilde{g}(\tilde{p} + a\tilde{\chi_i}) + \tilde{g}(\tilde{p} + b\tilde{\chi_j}) \geq \tilde{g}(\tilde{p}) + \tilde{g}(\tilde{p} + a\tilde{\chi_i} + b\tilde{\chi_j}) \tag{3.5}$$

holds for any $\tilde{p} = (p,q) \in \mathbb{R}^{N\cup S}$, distinct $i,j \in N \cup S$, and $a, b \geq 0$, where $\tilde{\chi_i}$ and $\tilde{\chi_j}$ are the $i$th and $j$th unit vectors in $\mathbb{R}^{N\cup S}$, respectively. For simplicity of notation we assume $\tilde{p} = \mathbf{0}$, and write $\tilde{p^i} = a\tilde{\chi_i}$, $\tilde{p^j} = b\tilde{\chi_j}$, and $\tilde{p^{ij}} = a\tilde{\chi_i} + b\tilde{\chi_j}$. Take $X, Y \subseteq N$ and $U, V \subseteq S$ such that $|X| + |U| = |Y| + |V| = r$,

$$\tilde{g}(\tilde{p}) = f(X) - \tilde{p}(X \cup U) = f(X),$$
$$\tilde{g}(\tilde{p^{ij}}) = f(Y) - \tilde{p^{ij}}(Y \cup V) = f(Y) - a|(Y \cup V) \cap \{i\}| - b|(Y \cup V) \cap \{j\}|.$$

Note also that $\tilde{g}(\tilde{p^i}) = \max\{f(Z) - a|(Z \cup W) \cap \{i\}| \mid Z \subseteq N,\ W \subseteq S\}$ and similarly for $\tilde{g}(\tilde{p^j})$.

If $\{i,j\} \subseteq N$, (3.5) reduces to $g(p + a\chi_i) + g(p + b\chi_j) \geq g(p) + g(p + a\chi_i + b\chi_j)$, which holds since $g$ is assumed to be submodular. The remaining cases are easier (not essential).

In case of $\{i,j\} \subseteq S$, we can assume, by $|S| \geq r - r' + 2$, that $V \cap \{i,j\} = \emptyset$, which implies that $\tilde{g}(\tilde{p^{ij}}) = g(p)$. Similarly, we have $\tilde{g}(\tilde{p^i}) = \tilde{g}(\tilde{p^j}) = g(p)$ as well as $\tilde{g}(\tilde{p}) = g(p)$. Therefore, (3.5) holds.

In case of $|N \cap \{i,j\}| = |S \cap \{i,j\}| = 1$, we may assume $i \in N$ and $j \in S$ by symmetry. By $|S| \geq r - r' + 2$, we have $\tilde{g}(\tilde{p^{ij}}) = \tilde{g}(\tilde{p^i}) = g(p^i)$ and $\tilde{g}(\tilde{p^j}) = \tilde{g}(\tilde{p}) = g(p)$, where $p^i = a\chi_i \in \mathbb{R}^N$ and $p^j = b\chi_j \in \mathbb{R}^N$. Therefore, (3.5) holds. □

We are now in the position to complete the proof of Theorem 1. If the conjugate function $g$ of $f$ is submodular, $\tilde{g}$ is also submodular by Lemma 9. Then $\tilde{f}$ is M-concave by Lemma 8, and therefore $f$ is M$^\natural$-concave by Proposition 3.

## 4. Appendix: Proof of Theorem 4

A self-contained proof of Theorem 4 is presented here. This is basically the same as the proof of Theorem 5.2.25 in [12, pp.295–297] adapted to our present notation, with the difference only in the proof of Claim 2.

Let $\mathcal{B} = \mathrm{dom}\, f$. For $p \in \mathbb{R}^N$ we define

$$f_p(X) = f(X) + p(X), \quad f_p(X,i,j) = f_p(X - i + j) - f_p(X) \qquad (X \in \mathcal{B}),$$

where $f_p(X,i,j) = -\infty$ if $X - i + j \notin \mathcal{B}$. For $X, Y \in \mathcal{B}$, $i \in X \setminus Y$, and $j \in Y \setminus X$, we have

$$f(X,i,j) + f(Y,j,i) = f_p(X,i,j) + f_p(Y,j,i). \tag{4.1}$$

If $X \in \mathcal{B}$, $X \setminus Y = \{i_0, i_1\}$, $Y \setminus X = \{j_0, j_1\}$ (with $i_0 \neq i_1$, $j_0 \neq j_1$), the local exchange property (condition (ii) in Theorem 4) implies‡

$$f_p(Y) - f_p(X) \leq \max\{f_p(X,i_0,j_0) + f_p(X,i_1,j_1),\ f_p(X,i_0,j_1) + f_p(X,i_1,j_0)\}. \tag{4.2}$$

‡If $Y \notin \mathcal{B}$, the inequality (4.2) is trivially true with $f_p(Y) = -\infty$.

Define

$$\mathcal{D} = \{(X,Y) \mid X, Y \in \mathcal{B},\ \exists\, i_* \in X \setminus Y,\ \forall\, j \in Y \setminus X : f(X) + f(Y) > f(X - i_* + j) + f(Y + i_* - j)\},$$

which denotes the set of pairs $(X, Y)$ for which the exchange property (2.1) fails. We want to show $\mathcal{D} = \emptyset$.

Suppose, to the contrary, that $\mathcal{D} \neq \emptyset$, and take $(X, Y) \in \mathcal{D}$ such that $|Y \setminus X|$ is minimum and let $i_* \in X \setminus Y$ be the element in the definition of $\mathcal{D}$. We have $|Y \setminus X| > 2$. Define $p \in \mathbb{R}^N$ by

$$p_j = \begin{cases} -f(X, i_*, j) & (j \in Y \setminus X,\ X - i_* + j \in \mathcal{B}), \\ f(Y, j, i_*) + \varepsilon & (j \in Y \setminus X,\ X - i_* + j \notin \mathcal{B},\ Y + i_* - j \in \mathcal{B}), \\ 0 & (\text{otherwise}) \end{cases}$$

with some $\varepsilon > 0$.

**Claim 1:**

$$f_p(X, i_*, j) = 0 \quad \text{if } j \in Y \setminus X,\ X - i_* + j \in \mathcal{B}, \tag{4.3}$$

$$f_p(Y, j, i_*) < 0 \quad \text{for } j \in Y \setminus X. \tag{4.4}$$

The inequality (4.4) can be shown as follows. If $X - i_* + j \in \mathcal{B}$, we have $f_p(X, i_*, j) = 0$ by (4.3) and

$$f_p(X, i_*, j) + f_p(Y, j, i_*) = f(X, i_*, j) + f(Y, j, i_*) < 0$$

by (4.1) and the definition of $i_*$. Otherwise we have $f_p(Y, j, i_*) = -\varepsilon$ or $-\infty$ according to whether $Y + i_* - j \in \mathcal{B}$ or not.

**Claim 2:** There exist $i_0 \in X \setminus Y$ and $j_0 \in Y \setminus X$ such that $i_0 \neq i_*$, $Y + i_0 - j_0 \in \mathcal{B}$, and

$$f_p(Y, j_0, i_0) \geq f_p(Y, j, i_0) \qquad (j \in Y \setminus X). \tag{4.5}$$

First, we show the existence of $i_0 \in X \setminus Y$ and $j \in Y \setminus X$ such that $Y + i_0 - j \in \mathcal{B}$ and $i_0 \neq i_*$. By connectedness of $\mathcal{B}$ and $|X \setminus Y| > 2$, there exist $i_1 \in X \setminus Y$ and $j_1 \in Y \setminus X$ such that $Z = Y + i_1 - j_1 \in \mathcal{B}$. If $i_1 \neq i_*$, we are done with $(i_0, j) = (i_1, j_1)$. Otherwise, again by connectedness, there exist $i_2 \in X \setminus Z$ and $j_2 \in Z \setminus X$ such that $W = Z + i_2 - j_2 \in \mathcal{B}$. Since $|W \setminus Y| = 2$ with $W = Y + \{i_1, i_2\} - \{j_1, j_2\}$, we obtain $Y + i_2 - j_1 \in \mathcal{B}$ or $Y + i_2 - j_2 \in \mathcal{B}$ from (2.1). Hence we can take $(i_0, j) = (i_2, j_1)$ or $(i_0, j) = (i_2, j_2)$; note that $i_2$ is distinct from $i_*$. Next we choose the element $j_0$. By the choice of $i_0$, we have $f_p(Y, j, i_0) > -\infty$ for some $j \in Y \setminus X$. By letting $j_0$ to be an element $j \in Y \setminus X$ that maximizes $f_p(Y, j, i_0)$, we obtain (4.5). Thus Claim 2 is established under the connectedness assumption.

**Claim 3:** $(X, Z) \in \mathcal{D}$ with $Z = Y + i_0 - j_0$.

To prove this it suffices to show

$$f_p(X, i_*, j) + f_p(Z, j, i_*) < 0 \qquad (j \in Z \setminus X).$$

We may restrict ourselves to $j$ with $X - i_* + j \in \mathcal{B}$, since otherwise the first term $f_p(X, i_*, j)$ is equal to $-\infty$. For such $j$ the first term is equal to zero by (4.3). For the second term it follows from (4.2), (4.4), and (4.5) that

$$\begin{aligned} f_p(Z, j, i_*) &= f_p(Y + \{i_0, i_*\} - \{j_0, j\}) - f_p(Y + i_0 - j_0) \\ &\leq \max\,[f_p(Y, j_0, i_0) + f_p(Y, j, i_*), f_p(Y, j, i_0) + f_p(Y, j_0, i_*)] - f_p(Y, j_0, i_0) \\ &< \max\,[f_p(Y, j_0, i_0), f_p(Y, j, i_0)] - f_p(Y, j_0, i_0) \ = 0. \end{aligned}$$

Since $|Z \setminus X| = |Y \setminus X| - 1$, Claim 3 contradicts our choice of $(X, Y) \in \mathcal{D}$. Therefore we conclude $\mathcal{D} = \emptyset$. This completes the proof of Theorem 4.

**Remark 4.1.** For the ease of reference, we describe here the necessary change in the proof of [12, Theorem 5.2.25] in the notation there. The necessary change is localized to the proof of

Claim 2: There exist $u_0 \in B \setminus B'$ and $v_0 \in B' \setminus B$ such that $u_0 \neq u_*$, $B' + u_0 - v_0 \in \mathcal{B}$,

$$\omega_p(B', v_0, u_0) \geq \omega_p(B', v, u_0) \qquad (v \in B' \setminus B). \tag{4.6}$$

We now assume connectedness of $\mathcal{B}$, instead of its exchange property. First, we show the existence of $u_0 \in B \setminus B'$ and $v \in B' \setminus B$ such that $B' + u_0 - v \in \mathcal{B}$ and $u_0 \neq u_*$. By connectedness of $\mathcal{B}$ and $|B \setminus B'| > 2$, there exist $u_1 \in B \setminus B'$ and $v_1 \in B' \setminus B$ such that $B'' = B' + u_1 - v_1 \in \mathcal{B}$. If $u_1 \neq u_*$, we are done with $(u_0, v) = (u_1, v_1)$. Otherwise, again by connectedness, there exist $u_2 \in B \setminus B''$ and $v_2 \in B'' \setminus B$ such that $B''' = B'' + u_2 - v_2 \in \mathcal{B}$. Since $|B''' \setminus B'| = 2$ with $B''' = B' + \{u_1, u_2\} - \{v_1, v_2\}$, we obtain $B' + u_2 - v_1 \in \mathcal{B}$ or $B' + u_2 - v_2 \in \mathcal{B}$ from (2.1). Hence we can take $(u_0, v) = (u_2, v_1)$ or $(u_0, v) = (u_2, v_2)$; note that $u_2$ is distinct from $u_*$. Next we choose the element $v_0$. By the choice of $u_0$, we have $\omega_p(B', v, u_0) > -\infty$ for some $v \in B' \setminus B$. By letting $v_0$ to be an element $v \in B' \setminus B$ that maximizes $\omega_p(B', v, u_0)$, we obtain (4.6). Thus Claim 2 is established under the connectedness assumption. ■

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