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CHOICE-BASED SEATING POSITION MODEL WITH UNDISTINGUISHED MULTI-LINES IN REVENUE MANAGEMENT

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Abstract In revenue management, there are models which aim to maximize revenue by controlling policy for uncertain demands throughout a booking horizon. The models are called dynamic models. One of the applications of the dynamic models is reservation system which offers available seats for customers' requests. Recently, the system has allowed us to choose our booking seat position. However, the dynamic models in revenue management have not been included customers' selection behavior for seating position. This paper proposes choice-based seating position model with undistinguished multi-lines that is a dynamic model considered with the customers' selection behavior for seating positions. Approximate solutions for this model are calculated by Choice-based Deterministic Linear Programming (CDLP) and decomposition approximation which are used in choice-based network revenue management models. This paper suggests that CDLP is more effective than decomposition approximation for the choice-based seating position model, even through some reports in revenue management suggested that decomposition approximation could derive higher revenue than CDLP in their models.

Keywords: Decision making, network revenue management, linear programming, seating position, choice behavior, multinomial logit choice model

1. Introduction

At facilities which have some features; fixed capacity and large fixed cost, the managers decide to accept or deny requests for perishable products. These scenes can be seen in several industries, for example, airline, hotel, rental car, opera, theater, and etc. The decision-making has been dealt with in revenue management (RM) [14]. In RM, there are mainly three kind of decision-makings, which are structural decisions, price decisions and quantity decisions. (See section 1.1 in [14].) To make the quantity decisions, there are dynamic models. The dynamic models are to find optimal policies under an assumption which is that requests of different segments arrive simultaneously throughout booking horizon. Lee and Hersh [8] presented single-leg models and its properties such as monotonicity. They assumed that booking requests arrive according to Poisson process, and also suggested how to discretize a booking horizon to approximate the arriving process. Subramanian et al. [13] included cancellation and overbooking in Lee and Hersh's models. El-Haber and El-Taha [5] extended the Subramanian et al.'s model to one with two types of seats, which means they connected the model with network RM (NRM).

NRM is a field of RM to simultaneously deal with multiple kinds of capacities. For example, a reservation system needs to handle multiple kinds of seats at the same time if the system offers transit tickets. (See section 3.1 in [14] for details.) For NRM, Chang et al. [4] reported recent applications and techniques.

When the transit tickets are offered, a customer often select buying the transit ticket or an alternative action which is for example to buy another ticket to go to he/her destination directly, or not to purchase any tickets. This customer's choice has already been considered in RM. Talluri and van Ryzin [14] suggested a single-leg model with customer's behavior, and showed some properties of the model. Gallego et al. [7] presented an RM model with flexible products and customers' behavior. They also suggested approximation method to calculate solutions by using linear programming, which is called choice-based deterministic linear programming method (CDLP). Zhang and Cooper [16] proposed a model for parallel flights with customers' behavior. Liu and van Ryzin [9] showed a basic NRM model with customers' behavior and they suggested decomposition approximation for their model. They presented that the decomposition approximation could make higher revenue than CDLP. Bront et al. [3] studied for Liu and van Ryzin's model. They revealed some structural properties and slightly enhanced the method. Recently, Sierag et al. [12] included overbooking and cancellation into the choice-based NRM model. Topics or models with customer's behavior in RM are summarized in [11] as overview.

In these RM models, state of reservation system is expressed as the number of booked capacities (or unbooked capacities). However, actual capacities, usually meaning seats, are often arranged in rows. For cases of theater or stadium, Talluri and van Ryzin [14] pointed out that there is a problem of isolated unbooked seats block. This means that groups or couples would not be willing to reserve seats if there are only separated seats, since they want to sit together. Ogasawara [10] focused on this problem and considered a dynamic model with seats placed on a single line. In [10], the system decides how to allocate positions to arriving groups. It implies that customers' choice behavior is not considered in the model. If customers can choose their booking seat positions as they want, then the behavior can generate undesirable state in which there are many isolated small unbooked seat blocks, as shown in Figure 1. In Figure 1, shaded cells and unshaded cells correspond to booked seats and unbooked seats, respectively. Many people do not apparently want to select any seat positions when they look this state to make reservations. Hence, we need to avert the undesirable state as much as possible to achieve expected maximum revenue.

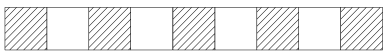

Figure 1: Isolated seats which are placed on a single line

In this paper, we propose a new dynamic model which is with customers' choice behavior for seating positions on multiple lines where the lines are not distinguished each other, and one seat links to one fare class. We call this model choice-based seating position model with undistinguished multi-lines. The situation can be seen in opera or Kabuki. For this model, it is hard to find optimal solutions if the number of lines or the number of seats of the longest line becomes large. However, if we apply Multinomial logit (MNL) choice model to customers' behavior, we can efficiently find approximate solutions in CDLP and decomposition approximation. In regard to exact optimal solutions, a range of searching for the solution in decision space can be reduced although the curse of dimensionality cannot be prevented. In numerical examples conducted in this paper, expected revenue derived from CDLP can be larger than one derived from decomposition approximation, which is different result from existing researches [3, 9]. If we take account of distinguished multi-lines, it is hard to compare approximate solutions to an exact solution because its state space becomes enormously larger than this our model's one. In addition, considering distinguished multi-lines gives rise to some variations of customers' choice behavior for seating positions on different lines and makes a model more complex. In this paper, we focus seating positions and suggest a model with customers' choice behavior that depends on a position relative to other booked seating positions.

The formulation for choice-based seating position model with undistinguished multi-lines is shown in section 2. In section 3, we present how to applying the approximation methods and MNL choice model to the choice-based seating position model. In section 4, we estimate solutions which are calculated by the approximation methods, by using some numerical examples.

2. Formulation

We consider a facility with multiple seat lines such as opera or Kabuki. Assume that these lines do not distinguished each other. In addition, a fare class links to a seat, and arrivals of requests are independent in different fare classes. From these assumptions, we can independently deal with the seats at each fare class and regard the numbers of different size of adjacent vacant seats as state of the system, using approach of the formulation in [10]. Therefore, we can consider the problem with only one fare class. In this paper, we treat cases with only one fare class. Let r be revenue which is generated from selling a seat of the fare class. Figure 2 shows the procedure of modifying a case with multiple lines and two fare classes as an example.

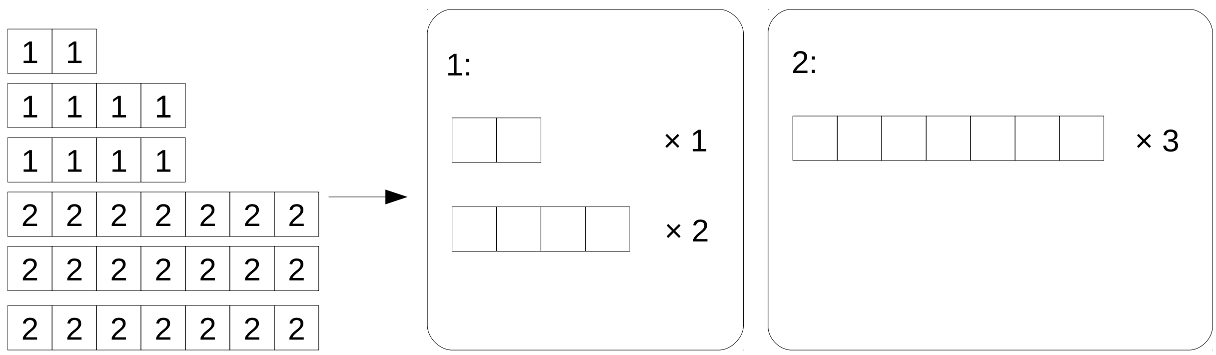

Figure 2: Changing multiple lines and two fare classes into multiple undistinguished lines with a single fare class

Booking horizon is sufficiently discretized into $t = 1, \cdots, T$ so that no more than one customer's request arrives at a time period. The time t indicates remaining time to the terminal time $t = 0$. Cancellation and overbooking are ignored in this paper. Let λ be arrival rate of the requests. States of the facility are shown by the numbers of adjacent vacant seats as we have already mentioned. We call a vacant seat block a *segment*. If the segment have n seats, then we call this the segment of size n. The left side and right side of segments are not distinguished. This assumption makes some undistinguished positions in a segment, which is shown in Figure 3 as an example. The positions connected by arrows show seats which are undistinguished each other in a segment of size 5.

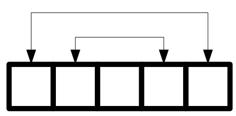

Figure 3: Undistinguished positions in a segment of size 5

Let m be the number of size of the longest segment in the facility. $c = (c_1, \cdots, c_m)^T$ stands for an initial state of the facility where T means transpose, and $c_i, i = 1, \cdots, m$

indicates the initial number of segments of size i. c is a state of the facility in a begging time T on the booking horizon. State space is defined as

$$X = \left\{ (x_1, \cdots, x_m)^T | 0 \le x_i \le \sum_{k=1}^{m} c_k \lfloor \frac{k+1}{i+1} \rfloor, x_i \in \mathbb{Z}_+, i = 1, \cdots, m \right\}$$

where $\lfloor \frac{k+1}{i+1} \rfloor$ means the number of segments of size i which is generated from a segment of size k. $\mathbb{Z}_+$ stands for the set of non-negative integer numbers. For example, one segment of size 3 produces $\lfloor \frac{3+1}{1+1} \rfloor = 2$ segments of size 1, as the following Figure 4.

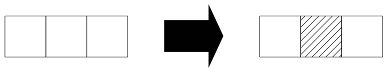

Figure 4: Separating a segment of size 3

In this model, if an arriving customer decides to book a seat, then he/she needs to decide which segment and which position to choose from available seats. Choices that a system of the facility can offer for customers at a state $x \in X$ are defined as below.

$$\Omega(x) = \left\{ (a, b) | x_a > 0, b \in \mathbb{Z}_+, 0 < b \le \frac{a+1}{2}, a = 1, \cdots, m \right\}, (x_i)_{i=1,\cdots,m} = x \in X$$

where a and b indicate the size of segment and an index of a seating position at the segment of size a, respectively. The empty set $\emptyset \subseteq \Omega(x), x \in X$ means not to offer any seating positions for arriving requests. In Figure 5, the elements of $\Omega(x'), x' = (0, 0, 0, 1, 1)^T$ are shown as an example.

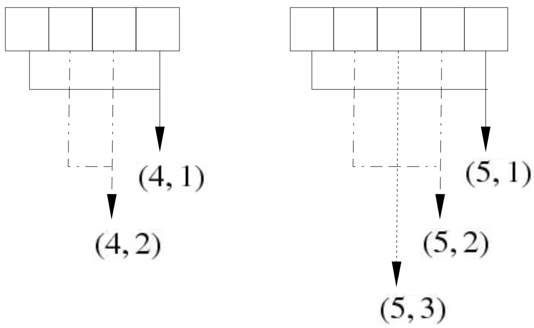

Figure 5: Elements of $\Omega(x'), x' = (0, 0, 0, 1, 1)^T$

For each $x \in X$, we call $\Omega(x)$ a possible choice set. The system of the facility decides a subset of the possible choice set $S \subseteq \Omega(x), x \in X$ on a beginning time point of each time duration for arriving customers. The subset S is called offer set. An arriving customer chooses his/her booking position from an offer set which is decided by the system of the facility to maximize expected revenue over booking horizon.

Customers' behavior in this model is discretely defined, which can also be seen in [3, 9, 14]. Arriving customers probabilistically choose their seating positions. Let $P_{(a,b)}(S)$ be a probability that an arriving customer selects the position $(a, b) \in S$ when $S \subseteq \Omega(x), x \in X$ is offered. Set $P_0(S) = 1 - \sum_{(a,b) \in S} P_{(a,b)}(S), S \subseteq \Omega(x), x \in X$ where $P_0(S)$ shows a probability of no-purchase when $S \subseteq \Omega(x), x \in X$ is offered.

Then, we consider changes of the numbers of segments if an arriving customer chooses $(a, b) \in S \subseteq \Omega(x), x \in X$. When the customer chooses the segment of size a and its index b, the change of state is $A_{(a,b)} = e_{b-1} + e_{a-b} - e_a$ where e_i is m-dimensional and i-th unit column vector for each $i = 1, \cdots, m$, and e_0 indicates zero vector. It means that the segment of size a is split into segments of size $b-1$ and $a-b$ by the customer's choice. In addition, we define the matrix $A = (A_{(a,b)})_{(a,b)\in\Omega}$ where

$$\Omega = \left\{ (a,b) | a = 1, \cdots, m, b \in \mathbb{Z}_+, 0 < b \leq \frac{a+1}{2} \right\}.$$

Ω means all possible offer sets that the system can decide for arriving customers over booking horizon. For instance, given $\Omega_3 = \{(3,1),(3,2)\}$ and $\Omega_4 = \{(4,1),(4,2)\}$, matrices A_3 and A_4 generated from Ω_3 and Ω_4, respectively, are

$$A_3 = (A_{(3,1)}, A_{(3,2)}) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ -1 & -1 \end{pmatrix} \text{ and } A_4 = (A_{(4,1)}, A_{(4,2)}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Using dynamic programming, let $U_t(x)$ be the maximum expected revenue which the facility can obtain by optimally operating from t to 0, given the state x in time t.

$$\begin{aligned} U_t(x) = \max_{S \subseteq \Omega(x)} \Bigg\{ & \lambda \sum_{(a,b)\in S} P_{(a,b)}(S)(r + U_{t-1}(x + A_{(a,b)})) \\ & + (\lambda P_0(S) + (1-\lambda))U_{t-1}(x) \Bigg\}, t = 1, \cdots, T, x \in X. \end{aligned} \tag{2.1}$$

Boundary conditions are $U_0(x) = 0, U_{T+1}(x) = 0, x \in X$, $U_t(\mathbf{0}) = 0$ and $U_t(x) = 0, x \notin X, t = 1, \cdots, T$. (2.1) is rewritten by $P_0(S) = 1 - \sum_{(a,b)\in S} P_{(a,b)}(S)$ as the following equations.

$$U_t(x) = \max_{S \subseteq \Omega(x)} \left\{ \lambda \sum_{(a,b)\in S} P_{(a,b)}(S)(r - \Delta_{(a,b)} U_{t-1}(x)) \right\} + U_{t-1}(x), \quad t = 1, \cdots, T, x \in X \tag{2.2}$$

where $\Delta_{(a,b)} U_t(x) = U_t(x) - U_t(x + A_{(a,b)})$. $\Delta_{(a,b)} U_t(x)$ means opportunity cost of the seat (a,b) at the state x in the time t.

In this paper, $|K|$ stands for the number of elements of the set K where the number of elements is finite.

We show optimal policies which are calculated from a modest numerical example. Let $N = 30, c = (0,0,1,2), r = 10$ and $\lambda = 0.3$. We assume that arrival customers have preference weights $v_{(a,b)}$ for seating positions (a,b). Let $v_{(1,1)} = 0.5, v_{(2,1)} = 1.5, v_{(3,1)} = 2.0, v_{(3,2)} = 3.0, v_{(4,1)} = 2.5, v_{(4,2)} = 3.5$ and $v_0 = 1.0$ where v_0 is a preference weight for no-choice. The customers probabilistically select their seating positions in accordance with ratios of the preference weights. In other words, customers' choices depend on a multinomial logit choice model. (The model is mentioned in 3.3.) Probabilities that a requests selects a seating position from an offer set $S' = \{(3,1),(4,1)\}$ are $P_{(3,1)}(S') = \frac{2.0}{2.0+2.5+1.0}, P_{(4,1)}(S') = \frac{2.5}{2.0+2.5+1.0}$ and $P_0(S') = \frac{2.0}{2.0+2.5+1.0}$ in this manner. Optimal offer sets for states $(0,0,1,2), (0,0,1,2), (1,1,1,1), (1,1,2,0)$ which are obtained from the numerical example are shown in Table 1.

Table 1: Optimal offer sets for states $(0, 0, 1, 2), (1, 1, 1, 1), (1, 1, 2, 0)$

$n \setminus x$	(0, 0, 1, 2)	(1, 1, 1, 1)	(1, 1, 2, 0)
1	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
2	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
3	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
4	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
5	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
6	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
7	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
8	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
9	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
10	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
11	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
12	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (3, 2), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
13	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1), (3, 2)}
14	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
15	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
16	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
17	{(3, 1), (3, 2), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
18	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
19	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
20	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
21	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
22	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
23	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
24	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
25	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
26	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
27	{(3, 1), (4, 1), (4, 2)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
28	{(3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
29	{(3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}
30	{(3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1), (4, 1)}	{(1, 1), (2, 1), (3, 1)}

In addition, we consider a sample path of state transition to show a control using this optimal offer sets. Suppose that a first request arrives in $n = 27$ and choices a position $(4, 2)$ from the optimal offer set $\{(3, 1), (4, 1), (4, 2)\}$, noting $c = (0, 0, 1, 2)$. Then, the initial state $(0, 0, 1, 2)$ transfers to $(1, 1, 1, 1)$. In $n = 22$, we postulate that a request arrives and choices $(4, 1)$ from the optimal offer set $\{(1, 1), (2, 1), (3, 1), (4, 1)\}$. This choice transfers state from $(1, 1, 1, 1)$ to $(1, 1, 2, 0)$. Figure 6 shows this control and path.

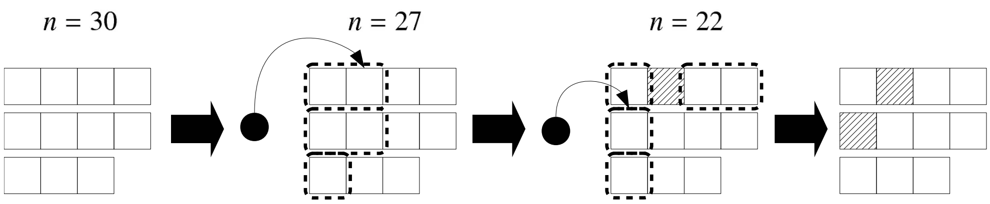

Figure 6: A sample path from $n = 30$ to $n = 22$

Then, we consider another case in which a system does not control seating positions for the earlier sample path, where what the system does not control seating positions means that all available seats are opened at all times. We call this control FULL-OPEN. (It is used again in section 4.) Under this control, an offer set for $n = 27$ is $\{(3, 1), (3, 2), (4, 1), (4, 2)\}$ in the earlier sample path. If the arriving request selects $(4, 2)$ from the offer set, then an offer set is $\{(1, 1), (2, 1), (3, 1), (4, 1), (4, 2)\}$ in $n = 22$. This difference of the offer sets in $n = 22$ between the optimal control and FULL-OPEN can make difference choice and difference path of state transition.

3. Approximation Methods

It is obviously difficult to compute the expected maximum revenue in (2.2) by the curse of dimensionality if m and the numbers of initial segments $c_i, i = 1, \cdots, m$ enlarge. Gallego et al. [7] suggested Choice-based Deterministic Linear Programming (CDLP) as an approximation methods in NRM with customer's choice behavior. Further, Liu and van Ryzin [9] suggested decomposition approximation.

In this section, we apply the CDLP and decomposition approximation to our model.

3.1. Choice-based deterministic linear programming

Let the numbers of segments (meaning capacity) be continuous, and arrival rates be deterministic number.

We set

$$R(S) = \sum_{(a,b)\in S} rP_{(a,b)}(S), S \subseteq \Omega(x), x \in X,$$

which is the expected revenue if S is offered. We define $P(S) = (P_{(a,b)}(S))^T_{(a,b)\in\Omega}$. If $P_{(3,1)}(S') = 0.25$ and $P_{(3,2)}(S') = 0.5$ are given for $\Omega_3 = \{(3,1),(3,2)\}$ where $S' = \{(3,1), (3,2)\}$, then $P(S') = (P_{(3,1)}(S'), P_{(3,2)}(S'))^T = (0.25, 0.5)^T$. In addition, let $Q(S)$ be the expected changing segment if offer set is S. Since the segment is continuous,

$$Q(S) = P(S)^T A^T$$

where $Q(S) = (Q_1(S), \cdots, Q_m(S))^T$, and $Q_i(S)$ means the expected changing number of the segment of size $i = 1, \cdots, m$ if S is offered.

If any subsets are generated from Ω, then impossible combinations of seating positions in offer sets on actual transition from a given initial state c would be created. Therefore, we make an assumption for the initial state c in this paper as below.

Assumption 3.1. On transition from a given initial state c over time horizon, there exists the states $x^+ = (x_1, \cdots, x_m)^T \in X$ such that $x_i > 0, i = 1, \cdots, m$.

We do not consider not to move to the states x^+ by deficiency of time because we identify that T is sufficiently large. We can naturally see whether c satisfies Assumption 3.1 if we trace all transitions of states. However, we can confirm whether c satisfies a sufficient condition for the assumption by Algorithm 3.1. The algorithm uses a simple fact which is that a segment β is increased by one at most if a segment $\alpha(> \beta)$ is consumed. The order of the algorithm is $O(m^2)$. In Algorithm 3.1, suppose that $m > 1$.

Algorithm 3.1 An algorithm to confirm whether c satisfies Assumption 3.1.

Input c
flag $\leftarrow$ False
for $i = 1$ to $m - 1$ **do**
  if there exists j such that $c_j \neq 0$, $i < j$ and $c_i = 0$, **then**
    $j^* \leftarrow \min\{j\}$
    $c_{j^*} \leftarrow c_{j^*} - 1$
  end if
end for
if $c_m \neq 0$ **then**
  flag $\leftarrow$ True
end if
Output flag

Let $t(S)$ be the total number of time periods in which S is offered. Then, we allow $t(S)$ to be continuous, which means that we can use the set S for some fraction of a time period. Note that the sequence to offer S is arbitrary.

Then, we obtain the maximum expected revenue U^{CDLP} by the method of CDLP.

$$\begin{aligned} U^{CDLP} = \max & \sum_{S\subseteq\Omega} \lambda R(S)t(S) \\ \text{s.t. } & 0 \le c + \sum_{S\subseteq\Omega} \lambda Q(S)t(S) \\ & \sum_{S\subseteq\Omega} t(S) \le T \\ & t(S) \ge 0, \forall S \subseteq \Omega. \end{aligned} \tag{3.1}$$

The first constraint equation in (3.1) indicates that elements of state $c_i, i = 1, \cdots, m$ do not become negative. The second constraint equation means that the total time allocating to all offer sets is not above T.

The dual problem of (3.1) is the following

$$\begin{aligned} \min\ & \pi^T c + T\sigma \\ \text{s.t. } & -\lambda\pi^T Q(S) + \sigma \ge \lambda R(S), \forall S \subseteq \Omega \\ & \pi \ge 0, \sigma \ge 0. \end{aligned} \tag{3.2}$$

π and σ are dual variables corresponding to the first and second constraint equations in (3.1), where $\pi = (\pi_1, \cdots, \pi_m)^T$. From sensitivity analysis (referring to [2]) for (3.2), we can see that π and σ mean an estimate of the marginal value of capacity on each segment and an estimate of the marginal value of time, respectively.

Similar to Liu and van Ryzin [9], we state that U^{CDLP} is an upper bound for the maximum expected revenue obtained from c and T. Define μ as control policy which maps states to control actions (offer sets). $S_\mu(t|\mathcal{F}_t)$ is an action in time t under the policy μ where $\mathcal{F}_t$ indicates the history of the system up to time t. To simplify notations, we omit $\mathcal{F}_t$ in the following sections. $N(S_\mu(t))$ denotes a $|\Omega|$-dimensional random vector which is the number of position purchased in time t under the policy μ. $N_{(a,b)}(S_\mu(t)) = 1, (a, b) \in \Omega$ means a sale of the position (a, b) and $N_{(a,b)}(S_\mu(t)) = 0, (a, b) \in \Omega$ means no sale of the the position (a, b). $\mathcal{M}$ denotes the class of all admissible policies. Noting that $\sum_{t=1}^{T} AN(S_\mu(t))$ is the changing quantities of the segments from T to 1,

$$0 \le c + \sum_{t=1}^{T} AN(S_\mu(t)) \quad \text{(a.s.)}$$

is satisfied.

From these notations, $U_T(c)$ is denoted as the form;

$$\begin{aligned} U^* = \max_{\mu\in\mathcal{M}} & E\left[r \sum_{t=1}^{T} e^T N(S_\mu(t)) \right] \\ \text{s.t. } & 0 \le c + \sum_{t=1}^{T} AN(S_\mu(t)) \quad \text{(a.s.)} \\ & S_\mu(t) \subseteq \Omega, t = 1, \cdots, T \end{aligned} \tag{3.3}$$

where e is a $|\Omega|-$dimensional column vector of which all elements are 1. We can obtain the following proposition from the similar way in [9].

Proposition 3.1. $U^* \leq U^{CDLP}$.

Proof. Let $S_{\mu^*}(t), t = 1, \cdots T$ be optimal controls. $0 \leq c + \sum_{t=1}^{T} AN(S_{\mu^*}(t))$ since μ^* is admissible policy. Then, set

$$t_{\mu^*}(S) = E\left[\sum_{t=1}^{T} 1_{S_{\mu^*}(t)}(S)\right]$$

and $t_{\mu^*}(S)$ is the expected total time in which S has been offered under the policy μ^*, where $1_{S_{\mu^*}(t)}(S), S \in \Omega$ is the indicator function. From Wald's equation (see p.521 in [6]),

$$\sum_{t=1}^{T} E\left[N_{(a,b)}(S_{\mu^*}(t))\right] = \sum_{S \subseteq \Omega} \lambda P_{(a,b)}(S) t_{\mu^*}(S)$$

and $0 \leq c + \sum_{S \subseteq \Omega} \lambda AP(S) t_{\mu^*}(S)$ are obtained. In addition, $U^* = \sum_{t=1}^{T} re^T E\left[N(S_{\mu^*}(t))\right] = \sum_{S \subseteq \Omega} \lambda re^T P(S) t_{\mu^*}(S)$. From definitions of $Q(S)$ and $R(S)$, we can find that $t_{\mu^*}(S)$ is a feasible solution for the problem (3.1). Hence, $U^* \leq U^{CDLP}$ is shown. □

We apply column generation (specifically cutting plane method seen in [2]) to (3.1), referring to [7, 9]. Let $\hat{S} \subseteq \Omega$. The reduced CDLP for the limited subset $\hat{S}$ is

$$\begin{aligned} U^{CDLP}(\hat{S}) = \max & \sum_{S \in \hat{S}} \lambda R(S) t(S) \\ \text{s.t.} \ & -\sum_{S \in \hat{S}} \lambda Q(S) t(S) \leq c \\ & \sum_{S \in \hat{S}} t(S) \leq T \\ & t(S) \geq 0, \forall S \in \hat{S} \end{aligned} \tag{3.4}$$

where π and σ are the optimal solutions of the dual problem for (3.4). These dual variables π and σ are corresponding to the first and second constraint equations in (3.4), respectively. We solve the following sub-problem to produce whether the dual solution is feasible for (3.1), or not.

$$\max_S \lambda(R(S) + \pi^T Q(S)) - \sigma. \tag{3.5}$$

If the optimal value of (3.5) is non-positive, then the dual solution is feasible and an optimal solution for (3.1). If the optimal value of (3.5) is positive, then we include the solution S^* to $\hat{S}$ and recalculate (3.4).

3.2. Decomposition approximation method

The solutions given by CDLP are times to allocate to each offer set. It means that a sequence of offering the each offer set through booking time is arbitrary. Hence, we cannot identify which offer set to apply for states and times even though we can see how much time to allocate to each offer set. To resolve this problem of CDLP, Liu and van Ryzin [9] suggested decomposition approximation for choice-based NRM models. In the rest of this paper, we apply a decomposition approximation improved by Bront et al. [3] to our model.

In the decomposition approximation, we decompose (2.2) in each segment by using a marginal value which is given as the dual solution $\pi = (\pi_1, \cdots, \pi_m)^T$, that is,

$$U_t(x) \approx \hat{U}_t^i(x_i) + \sum_{l \neq i} \pi_l x_l, t = 1, \cdots, T, i = 1, \cdots, m, x = (x_1, \cdots, x_m)^T \in X. \quad (3.6)$$

Phase 1: calculate one-dimensional dynamic programming

We calculate the one-dimensional dynamic programming $\hat{U}_t^i(x_i), t = 1, \cdots, T, i = 1, \ldots, m$, $x = (x_1, \cdots, x_m)^T \in X, (a, b) \in S \subseteq \Omega(x)$.

From (3.6),

$$\Delta_{(a,b)} U_t(x) = U_t(x) - U_t(x + A_{(a,b)}) \approx \hat{U}_t^i(x_i) - \hat{U}_t^i(x_i + e_i^T A_{(a,b)}) - (\pi^T - \pi_i e_i^T) A_{(a,b)}, \quad (3.7)$$

$$t = 1, \cdots, T, i = 1, \cdots, m, x \in X, (a, b) \in S \subseteq \Omega(x)$$

is obtained. Using (3.6) and (3.7),

$$\hat{U}_t^i(x_i) = \max_{S \in \overline{S}_i(x_i)} \left\{ \lambda \sum_{(a,b) \in S} P_{(a,b)}(S)(r + (\pi^T - \pi_i e_i^T) A_{(a,b)} - \Delta_{(a,b)} \hat{U}_{t-1}^i(x_i)) \right\} + \hat{U}_{t-1}^i(x_i) \quad (3.8)$$

is produced from (2.2) where $\overline{S}_j(x_j) = \{S | S \subseteq \Omega(x), x_i = x_j, x \in X\}$. $\overline{S}_j(x_j)$ is an decision space if the value of the i-th element of the state is x_j. Boundary conditions are $\hat{U}_0^i(x_i) = 0, \hat{U}_{T+1}^i(x_i) = 0, 0 \leq x_i \leq \sum_{k=1}^m c_k \lfloor \frac{k+1}{i+1} \rfloor, i = 1, \cdots, m$ and $\hat{U}_t^i(x_i) = 0, x_i < 0, \sum_{k=1}^m c_k \lfloor \frac{k+1}{i+1} \rfloor < x_i, i = 1, \cdots, m, t = 1, \cdots, T$.

Phase 2: find offer sets

We approximately calculate the deflection vector $\Delta U_t^i(x) = U_t(x) - U_t(x - e_i)$ for each $i = 1, \cdots, m$, $t = 1, \cdots, T$, and $x \in X$ by using $\hat{U}_t^i(x_i)$ in Phase 1. From the deflection vectors, we find offer sets for each time $t = 1, \cdots, T$ and state $x \in X$.

We consider a heuristic parameter $0 \leq \beta \leq 1$ and calculate

$$\Delta U_t^i(x) \approx \Delta \overline{U}_t^i(x) = \beta \Delta \hat{U}_t^i(x_i) + (1 - \beta)\pi_i, i = 1, \cdots, m, t = 1, \cdots, T, x \in X \quad (3.9)$$

where $\Delta \hat{U}_t^i(x_i) = \hat{U}_t^i(x_i) - \hat{U}_t^i(x_i - 1)$ and $\Delta \hat{U}_t^i(0) = 0$. Using $\Delta \hat{U}_t^i(x)$, for each time $t = 1, \cdots, T$ and state $x \in X$, solve

$$\max_{S \subseteq \Omega(x)} \left\{ \lambda \sum_{(a,b) \in S} P_{(a,b)}(S)(r - \Delta \overline{U}_{t-1}(x) A_{(a,b)}) \right\} \quad (3.10)$$

where $\Delta \overline{U}_t(x) = (\Delta \overline{U}_t^1(x), \cdots, \Delta \overline{U}_t^m(x))$. Thus, offer sets for each time and state can be obtained by the decomposition approximation.

3.3. Applying MNL choice model to customers' behavior

Remark that these approximation methods work for the curse of dimensionality. Specifically, the approach of CDLP is to resolve it by sacrificing sequences of offer sets, and the approach of decomposition approximation is to reduce it by decomposing state space. However, these methods do not resolve enlarging size of possible offer set. The number of seats of a fare class in stadium, opera, Kabuki and etc. is actually very large. The dimension m is derived from the number of seats at the longest line. Since the number of possible offer set is considerably effected by the dimension, it is obvious that the number of all subsets of Ω is exponentially increased. In the approximation methods, this problem appears in calculating (3.5), (3.8) and (3.10).

We show to able to resolve this problem under an assumption which is that customers' behavior depends on Multinomial Logit (MNL) choice model, which can be also seen in [7, 9]. This assumption means that customers have preferences (weights) for each position, and select a position by their preferences. Then, let $y = (y_{(a,b)})_{(a,b)\in\Omega}$ be a binary vector where

$$y_{(a,b)} = \begin{cases} 1 & (a,b) \text{ is available,} \\ 0 & (a,b) \text{ is non-available.} \end{cases}$$

For example, if seating positions $(3,1)$, $(3,2)$ and $(4,1)$ are available for $\Omega' = \{(3,1),(3,2),(4,1),(4,2)\}$, then a binary vector y' for the situation is $y' = (y_{(3,1)}, y_{(3,2)}, y_{(4,1)}, y_{(4,2)}) = (1,1,1,0)$.

Using y, we define

$$P_{(a,b)}(y) = \frac{v_{(a,b)}y_{(a,b)}}{\sum_{(\alpha,\beta)\in\Omega} v_{(\alpha,\beta)}y_{(\alpha,\beta)} + v_0} \tag{3.11}$$

as a probability that a customer chooses the position $(a,b) \in \Omega$, where $v_{(a,b)}(\geq 0)$ and $v_0(> 0)$ are preferences of purchasing the position $(a,b) \in \Omega$ and a preference of no-purchase, respectively. Applying the MNL choice model to (2.2), (3.5) and (3.10), we obtain

$$U_t(x) = \max_{y\in Y(x)} \frac{\lambda \sum_{(a,b)\in\Omega}(r - \Delta_{(a,b)}U_{t-1}(x))v_{(a,b)}y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)}y_{(a,b)} + v_0} + U_{t-1}(x),$$
$$t = 1, \cdots, T, x \in X, \tag{3.12}$$

$$\lambda \max_{y\in\{0,1\}^{|\Omega|}} \frac{\sum_{(a,b)\in\Omega}(r + \pi^T A_{(a,b)})v_{(a,b)}y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)}y_{(a,b)} + v_0} - \sigma \tag{3.13}$$

and

$$\max_{y\in Y(x)} \left\{ \frac{\lambda \sum_{(a,b)\in\Omega}(r - \Delta_{(a,b)}\hat{U}_{t-1}(x))v_{(a,b)}y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)}y_{(a,b)} + v_0} \right\}, x \in X, \tag{3.14}$$

respectively, where

$$Y(x) = \left\{(y_{(a,b)})_{(a,b)\in\Omega} | y_{(a,b)} \in \{0, 1_{\Omega(x)}(a,b)\}, (a,b) \in \Omega\right\}, x \in X. \tag{3.15}$$

We cite Proposition 6 in [9] as the following Proposition 3.2 which is arranged for the model of this paper.

Proposition 3.2 (Liu and van Ryzin [9])**.** Consider a problem

$$\max_{y\in\{0,1\}^{|\Omega|}} \frac{\sum_{(a,b)\in\Omega} \xi_{(a,b)} v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0}. \tag{3.16}$$

When $\xi_{(a,b)}, (a,b) \in \Omega$ are ranked in a decreasing order, let $\xi_{[i]}$ be the i-th value, that is,

$$\xi_{[1]} \geq \cdots \geq \xi_{[i]} \geq \cdots \geq \xi_{[|\Omega|]}.$$

Then, there is a critical value $k^*, 1 \leq k^* \leq |\Omega|$ such that

$$y^*_{(a,b)} = \begin{cases} 1 & \xi_{(a,b)} \geq \xi_{[k^*]}, \\ 0 & \xi_{(a,b)} < \xi_{[k^*]} \end{cases}$$

where $y^* = (y^*_{(a,b)})_{(a,b)\in\Omega}$ is an optimal solution for the above problem (3.16).

We show Proposition 3.3 which is easily obtained from Proposition 3.2.

Proposition 3.3. Given a state $x \in X$, consider the problem

$$\max_{y\in Y(x)} \frac{\sum_{(a,b)\in\Omega} \xi_{(a,b)} v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0}. \tag{3.17}$$

When $\xi_{(a,b)}, (a,b) \in \Omega(x)$ are ranked in a decreasing order, let $\xi_{[i]}$ be the i-th value, that is,

$$\xi_{[1]} \geq \cdots \geq \xi_{[i]} \geq \cdots \geq \xi_{[|\Omega(x)|]}.$$

Then, there is a critical value $k^*, 1 \leq k^* \leq |\Omega|$ such that

$$y^*_{(a,b)} = \begin{cases} 1 & \xi_{(a,b)} \geq \xi_{[k^*]}, 1_{\Omega(x)}(a,b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $y^* = (y^*_{(a,b)})_{(a,b)\in\Omega}$ is an optimal solution for the above problem (3.17).

Proof. Consider the problem

$$\max_{y\in Y'(x)} \frac{\sum_{(a,b)\in\Omega(x)} \xi_{(a,b)} v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega(x)} v_{(a,b)} y_{(a,b)} + v_0} \tag{3.18}$$

where

$$Y'(x) = \left\{(y_{(a,b)})_{(a,b)\in\Omega(x)} | y_{(a,b)} \in \{0,1\}, (a,b) \in \Omega(x)\right\}, x \in X. \tag{3.19}$$

It is obvious that the problem (3.17) is the same as the problem (3.18) because terms of (a,b) such that $1_{\Omega(x)}(a,b) = 0$ is zero in (3.17).

Therefore, Proposition 3.3 is obtained by applying Proposition 3.2 to the problem (3.18). □

From Proposition 3.2, we can solve (3.13) by calculating $|\Omega|$ patterns at most. This application for sub-problem of CDLP is similar to a result in [9]. To solve (3.12) and (3.14), we can use Proposition 3.3.

4. Numerical Examples

We estimate the each approximation method by Monte Carlo simulation. In regard to time, arrival rate and revenue, we set $T = 100, \lambda = 0.3$ and $r = 10$. We set five cases for initial states, which are $c = (0, 0, k, k), k = 3, 4, 5, 6, 7$. Let customers' preferences be $v_{(1,1)} = 0.5, v_{(2,1)} = 1.5, v_{(3,1)} = 2.0, v_{(3,2)} = 3.0, v_{(4,1)} = 2.5$ and $v_{(4,2)} = 3.5$. In regard to a preference of no-purchase which indicates a flow of paths, we set four cases $v_0 = 1, 2, 3$ and 4. Configurations of the approximation methods and policies are suggested as below.

DP: DP means that using offer sets which are obtained by (3.12).

CDLP-LX: Using solutions obtained by CDLP, we allocate offer sets with positive allocating time to each occurred state in backward lexicographical order. The backward lexicographical order means that we allocate offer sets to each occurred state in backward order when we regard the offer sets as character strings and sort the strings in lexicographical order, where positions in each offer set have been sorted in lexicographical order. For example, when there are offer sets $\{(1, 1), (3, 1)\}$ and $\{(2, 1), (3, 1), (3, 2)\}$ with positive allocating time, we select $\{(2, 1), (3, 1), (3, 2)\}$ under this rule.

CDLP-RND: Using solutions obtained by CDLP, we randomly choose an offer set from offer sets with positive allocating time.

DCOMP-0, DCOMP-0.5, DCOMP-1: DCOMP-0, DCOMP-0.5 and DCOMP-1 correspond to use offer sets which are computed by the decomposition approximation with $\beta = 0$, $\beta = 0.5$ and $\beta = 1$, respectively.

FULL-OPEN: This indicates that we always open all unbooked seats.

We trace paths of state from $n = N$ to $n = 0$ by 20000 times under the each above policy, and we calculate averages of total revenue obtained for each history and the policy. The results are shown in Table 2 where the numbers in the cells indicate percents of averages obtained by the configurations without FULL-OPEN to averages obtained by FULL-OPEN. Remark that FULL-OPEN actually does not control offer set for seating position.

It is natural that DP is the highest method than others. In several cases, higher revenue than FULL-OPEN is generated by CDLP in which the curse of dimensionality and increment of search range in decision space are resolved. Furthermore, we cannot find what one of the two configurations for CDLP; LX and RND is the better one. On the other hand, DCOMP does not derive higher revenue than FULL-OPEN in all input data-set and all patterns of the heuristic parameter. From these computations, we can suggest that decomposition approximation may not be effective one for this model. This result for DCOMP cannot be seen in [3, 9] in which they suggested that DCOMP substantially generated higher revenue than CDLP.

Note that there exists no difference among the all configurations if $v_0 = 0$. The reason is that it is optimal to accept all requests if all arriving customers always book any positions, since revenue obtained from each customer is the same in this model. Therefore, the difference is generated from customers' behavior, that is, we can obtain higher revenue by controlling choices of seating position for arriving customers even though fares for all seats are the same.

Table 2: The calculated percents for each configuration

	v_0	DP	CDLP LX	CDLP RND	DCOMP 0	DCOMP 0.5	DCOMP 1
(0,0,3,3)	1	111.99	64.93	79.79	87.63	94.87	93.75
	2	116.66	89.27	90.71	91.27	92.37	93.96
	3	115.89	96.34	99.02	93.54	93.44	91.69
	4	114.51	100.98	95.94	98.20	93.75	89.67
(0,0,4,4)	1	113.61	102.12	100.71	85.15	92.52	86.67
	2	113.73	104.01	100.41	97.10	91.34	83.49
	3	111.56	100.08	95.49	95.20	92.73	83.80
	4	109.13	98.93	95.27	97.12	93.70	85.09
(0,0,5,5)	1	110.53	103.54	100.76	97.37	87.40	85.82
	2	108.87	101.55	98.27	95.45	87.41	78.86
	3	106.99	95.35	98.35	93.34	90.38	79.21
	4	105.69	97.26	99.04	96.05	92.03	79.58
(0,0,6,6)	1	106.72	101.15	101.34	96.83	84.37	82.10
	2	105.30	97.65	99.22	93.79	84.18	74.67
	3	104.29	102.58	102.58	99.90	77.06	68.05
	4	103.67	100.41	100.12	99.95	79.88	67.07
(0,0,7,7)	1	103.56	100.93	101.07	95.64	81.53	79.34
	2	103.38	101.65	101.37	99.95	73.86	69.18
	3	102.77	99.39	99.60	99.99	74.28	64.67
	4	102.01	96.87	97.07	99.98	76.86	63.79

5. Conclusion and Future Issues

We presented choice-based seating position model with undistinguished multi-lines which deals with customers who choose their booking position or do not purchase seats on their preferences. If customers' behavior depends on MNL choice model, we can efficiently calculate offer sets by Propositions 3.2 and 3.3. For the model by dynamic programming, we can reduce searching in decision space and obtain the maximum expected revenue although the curse of dimensionality is not resolved. For CDLP method, we can efficiently solve its sub-problem even if the input parameters, m, $c_i, i = 1, \cdots, m$ and T enlarge. Further, it is suggested that CDLP might be effective approximation methods for the choice-based seating position model since there are some cases in which offer sets obtained by CDLP can generate higher revenue than FULL-OPEN. On the other hand, in the decomposition approximation method, a part of procedures for finding approximate solutions cannot be effectively reduced. In addition, the decomposition approximation did not work well in our all numerical examples. We can guess that this unsuccessful might be caused by the strong relationship among segments in state space of this model, unlike other RM models.

For the choice-based seating position model that has been stated in this paper, some extensions can be considered. For instance, it is to take account of multiple customer's segments with different preferences (e.g. membership or non-membership who has different enthusiasm for services provided by facilities). Other extension is to include multiple fare classes and distinguished multiple lines. These future issues come from a physical feature of seats which is that the seats are placed in rows.

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