FUZZY DECISION PROCESSES: FORMULATION AND ITS OPTIMALITY

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1. Introduction

Bellman and Zadeh [1] considered a sequence of fuzzy sets in a finite state space by solving the fuzzy relational equation written in the matrix form and obtained a maximizing decision for fuzzy multistage decision processes with a defined terminal time.

This paper gives an overview of a dynamic fuzzy system, which has been developed by the authors, and we discuss additional considerations as a new multi-stage fuzzy decision processes. So the details for the proof of theorems are referred to the cited papers.

In Section 2 a dynamic fuzzy system is formulated explicitly and the limit theorem for the sequence of states is shown by a contractive assumption. In Section 3,4 we will develop a new optimization problem including the discounted fuzzy reward and the time average fuzzy reward. The optimality is discussed under a partial order, which is called a fuzzy max order. For both performance indexes, we consider the fuzzy relational equation which is satisfied by the optimal policy. Further the policy improvement and the optimality equation are considered, which are the fundamental elements in Markov decision processes. In Section 5 we also discuss several subjects which should be analyzed in the future.

2. Sequences of Fuzzy states and the limit theorem

Firstly we begin with making a formulation for a dynamic fuzzy system. Let $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ and E be a compact metric space. We denote by $\mathcal{C}(E)$ the collection of all the closed subsets of E. Let ρ be the Hausdorff metric on $\mathcal{C}(E)$. Then it is well-known [2] that $(\mathcal{C}(E), \rho)$ is a compact metric space.

Throughout this paper, we define a fuzzy set on E by its membership function $\tilde{s} : E \mapsto [0,1]$. For the theory of fuzzy sets, we refer to Zadeh [21] and Novák [12]. Let $\mathcal{F}(E)$ be the set of all the fuzzy sets $\tilde{s}$ on E which are upper semi-continuous and satisfy $\sup_{x\in E} \tilde{s}(x) = 1$. For any fuzzy state $\tilde{s} \in \mathcal{F}(E)$ and $\alpha\,(0 \le \alpha \le 1)$, the α-cut is defined as follows:

$$\tilde{s}_\alpha := \{x \in E \mid \tilde{s}(x) \ge \alpha\}\ (\alpha > 0), \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in E \mid \tilde{s}(x) > 0\}$$

where cl means the closure of a set.

From the dynamic fuzzy system, we can define the sequence of fuzzy states $\{\tilde{s}_t\}_{t=0}^{\infty}$ by

$$\tilde{s}_0 = \tilde{s}, \quad \tilde{s}_{t+1}(y) = \sup_{x\in E}\{\tilde{s}_t(x) \wedge \tilde{q}(x,y)\}, \quad y \in E,\ t \ge 0. \tag{2.1}$$

The transition from $\tilde{s}_t$ to $\tilde{s}_{t+1}$ in (2.1) is called a fuzzy transition in the dynamic system.

Definition 2.1 (see [11]). For $\tilde{s}_t, \tilde{s} \in \mathcal{F}(E)$,

$$\lim_{t\to\infty} \tilde{s}_t = \tilde{s} \overset{\mathrm{Def}}{\Longleftrightarrow} \sup_{\alpha\in[0,1]} \rho(\tilde{s}_{t,\alpha}, \tilde{s}_\alpha) \to 0 \quad (t \to \infty)$$

provided that $\tilde{s}_{t,\alpha}, \tilde{s}_\alpha$ are α-cuts $(0 \le \alpha \le 1)$ for the fuzzy states $\tilde{s}_t, \tilde{s}$ respectively and ρ is the given Hausdorff metric. For $\alpha \in [0,1], D \in \mathcal{C}(E)$, we define a map $\tilde{q}_\alpha : \mathcal{C}(E) \mapsto \mathcal{C}(E)$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{y \in E \mid \tilde{q}(x,y) \ge \alpha \ \text{ for some } x \in D\} & \text{for } \alpha > 0, D \ne \emptyset, \\ \mathrm{cl}\{y \in E \mid \tilde{q}(x,y) > 0 \ \text{ for some } x \in D\} & \text{for } \alpha = 0, D \ne \emptyset, \\ E & \text{for } D = \emptyset. \end{cases} \tag{2.2}$$

Further, for $t = 0, 1, 2, \cdots$, we define maps $\tilde{q}_\alpha^t : \mathcal{C}(E) \mapsto \mathcal{C}(E)$ by

$$\tilde{q}_\alpha^0 \text{ is an identity map} \quad \text{and} \quad \tilde{q}_\alpha^t = \tilde{q}_\alpha\,(\tilde{q}_\alpha^{t-1})\ (t = 1, 2, \cdots). \tag{2.3}$$

The fuzzy transition is characterized by the followings [6].

Lemma 2.1. *For $\tilde{s} \in \mathcal{F}(E)$, it holds that*

$$\tilde{s}_{t,\alpha} = \tilde{q}^t_\alpha(\tilde{s}_\alpha) \quad \textit{for all } t = 0, 1, 2, \cdots \textit{ and } \alpha \in [0,1],$$

Assumption 2.1 (Contraction property). There exists a real number β $(0 < \beta < 1)$ satisfying the following condition :

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta\ \rho(A, B) \quad \text{for all } A, B \in \mathcal{C}(E) \text{ and all } \alpha\ (0 \leq \alpha \leq 1).$$

Theorem 2.1.

(i) *There exists a unique fuzzy state $\tilde{p} \in \mathcal{F}(E)$ satisfying*

$$\tilde{p}(y) = \max_{x \in E}\{\tilde{p}(x) \wedge \tilde{q}(x, y)\} \quad \textit{for all } y \in E. \tag{2.4}$$

(ii) *The sequence $\{\tilde{s}_t\}$ defined by (2.1) converges to a unique solution $\tilde{p}$ of (2.4) independently of the initial fuzzy state $\tilde{s}_0$. Namely,*

$$\lim_{t \to \infty} \tilde{s}_t = \tilde{p}. \tag{2.5}$$

In [17, 19] the monotonicity is imposed on the fuzzy relation $\tilde{q}$. They discussed the convergence and the recurrence for $\{\tilde{s}_t\}$. If the linearity structure of $\tilde{q}$ is assumed, the existence of potential associated with the fuzzy relation could be shown(cf. [14, 15, 16]).

As an example for the previous results, the next simple one is illustrated. Let $S = [0, 1]$ be a space of states and a fuzzy relation is given by

$$\tilde{q}(x, y) = 1 - |y - (x/2 + 1/4)|, \quad x, y \in S,$$

and an initial fuzzy state $\tilde{s}_0(x) = 1 - 2|x - 1/2|, x \in S$. Then we can easily check that the contraction coefficient β in Assumption 2.1 is 1/2 and that the sequence of fuzzy states defined by (1.2) is $\tilde{s}_n(x) = 1 - a_n |x - 1/2|, \quad x \in S$, where $a_0 = 2$ and $a_n = 2a_{n-1}/(2a_{n-1} + 1)$. Then, the limit fuzzy set $\tilde{p}$ of $\tilde{s}_n$ is

$$\tilde{p}(x) = 1 - 1/2\,|x - 1/2|, \quad x \in S,$$

which is also the unique solution of Theorem 2.1. Figure 1 shows the fuzzy relation $\tilde{q}(x, y)$ and Figure 2 shows the sequence of fuzzy states $\tilde{s}_n(x), n \geq 0$, which converges to $\tilde{p}(x)$.

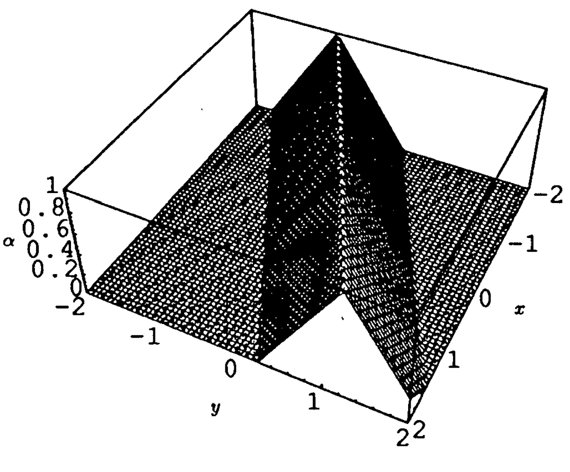

Fig. 1 : The fuzzy relation $\tilde{q}(x, y)$.

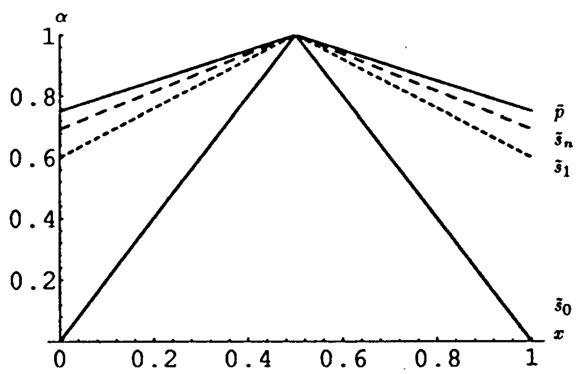

Fig. 2 : The sequence of $\tilde{s}_n$ and the solution $\tilde{p}$.

3. Fuzzy decision processes with a discounted total reward

We give notations and mathematical facts in order to formulate a fuzzy decision processes. Let E, E_1, E_2 be convex compact subsets of some Banach space. Denote by $\mathcal{F}(E)$ the set of all fuzzy sets $\tilde{s}$ on E which is upper semi-continuous and have a compact support with the normality condition: $\sup_{x\in E}\tilde{s}(x)=1$. The fuzzy relation between the space E_1 and E_2 means that $\tilde{q}: E_1 \times E_2 \to [0,1]$ and $\tilde{q} \in \mathcal{F}(E_1 \times E_2)$. A fuzzy relation $\tilde{q} \in \mathcal{F}(E_1 \times E_2)$ is called convex if

$$\tilde{q}(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \geq \tilde{q}(x_1, y_1) \wedge \tilde{q}(x_2, y_2) \tag{3.1}$$

for $x_1, x_2 \in E_1, y_1, y_2 \in E_2$, and $\lambda \in [0,1]$. The class of all convex fuzzy set is denoted by

$$\mathcal{F}_c(E) := \{\tilde{s} \in \mathcal{F}(E) \mid \tilde{s} \text{ is convex }\}.$$

Hereafter the set of all non-empty convex closed subset of E is denoted by $\mathcal{C}(E)$.

Especially when $E=[0,M]$ (a closed interval with a fixed $M>0$), we write

$$\mathcal{F}_c([0,M]) := \{\tilde{s} \in \mathcal{F}_c(R_+) \mid \tilde{s}_0 \subset [0,M]\},$$

and $\mathcal{C}([0,M])$ becomes the set of all non-empty closed sub-intervals of $[0,M]$.

In the rest of this talk, we consider Markov-type fuzzy decision processes

$$(S, A, [0,M], \tilde{q}, \tilde{r}, \beta)$$

which satisfy the following (M.i) — (M.iii):

(M.i) Let S and A be a state space and an action space, which are given as convex compact subsets of some Banach space respectively. The decision process is assumed to be fuzzy itself, so that both the state of the system and the action taken at each step are denoted by elements of $\mathcal{F}_c(S)$ and $\mathcal{F}_c(A)$, called the fuzzy state and the fuzzy action respectively.

(M.ii) The law of motion for the system and the fuzzy reward can be characterized by time invariant fuzzy relations $\tilde{q} \in \mathcal{F}_c((S \times A) \times S)$ and $\tilde{r} \in \mathcal{F}_c((S \times A) \times [0,M])$, where M is a given positive number. Explicitly, if the system is in a fuzzy state $\tilde{s} \in \mathcal{F}_c(S)$ and the fuzzy action $\tilde{a} \in \mathcal{F}_c(A)$ is chosen, then it transfers to a new fuzzy state $Q(\tilde{s},\tilde{a})$ and a fuzzy reward $R(\tilde{s},\tilde{a})$ has been incurred, where Q, R are defined by the following:

$$Q(\tilde{s},\tilde{a})(y) := \sup_{(x,a)\in S\times A} \tilde{s}(x) \wedge \tilde{a}(a) \wedge \tilde{q}(x,a,y) \quad (y \in S) \tag{3.2}$$

and

$$R(\tilde{s},\tilde{a})(u) := \sup_{(x,a)\in S\times A} \tilde{s}(x) \wedge \tilde{a}(a) \wedge \tilde{r}(x,a,u) \quad (0 \leq u \leq M). \tag{3.3}$$

(M.iii) The constant scalar β is a discount rate satisfying $0 < \beta < 1$.

Firstly we define a policy based on the fuzzy state and fuzzy action as follows. Let $\Pi := \{\pi \mid \pi : \mathcal{F}_c(S) \mapsto \mathcal{F}_c(A)\}$ be a set of all maps from $\mathcal{F}_c(S)$ to $\mathcal{F}_c(A)$. Any element $\pi \in \Pi$ is called a strategy. A policy, $\tilde{\pi} = (\pi_1, \pi_2, \pi_3, \cdots)$, is a sequence of strategies such that $\pi_t \in \Pi$ for each t. Especially, the policy $(\pi, \pi, \pi, \cdots)$ is a stationary policy and denoted by π^∞.

For any policy $\tilde{\pi} = (\pi_1, \pi_2, \cdots)$ and any initial fuzzy state $\tilde{s} \in \mathcal{F}_c(S)$, we define a sequence of fuzzy states $\{\tilde{s}_t\}_{t=0}^{\infty}$ as

$$\tilde{s}_1 := \tilde{s}, \quad \tilde{s}_{t+1} := Q(\tilde{s}_t, \pi_t(\tilde{s}_t)) \quad \text{for} \quad t = 1, 2, \cdots \tag{3.4}$$

For fuzzy sets $\tilde{m}, \tilde{n} \in \mathcal{F}_c(\mathbb{R}_+)$ and a scalar λ,

$$\begin{aligned} (\tilde{m}+\tilde{n})(x) &:= \sup_{x=y+z} \{\tilde{m}(y) \wedge \tilde{n}(z)\}, \\ (\lambda\tilde{m})(x) &:= \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0, \\ 1_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \quad x \in \mathbb{R}_+, \end{aligned} \tag{3.5}$$

where $1_A(\cdot)$ is the classical characteristic function for any ordinary subset A of $\mathbf{R}^n_+$.

Using the above operations, we can define the *discounted total fuzzy reward* as follows:

$$\psi(\tilde{\pi}, \tilde{s}) := \sum_{t=1}^{\infty} \beta^{t-1} R(\tilde{s}_t, \pi_t(\tilde{s}_t)) \in \mathcal{F}_c([0, M/(1-\beta)]) \tag{3.6}$$

for $\tilde{s} \in \mathcal{F}_c(S)$ and $\tilde{\pi} = (\pi_1, \pi_2, \cdots)$.

Definition 3.2 (Fuzzy max order [5]) For $\tilde{n}, \tilde{m} \in \mathcal{F}_c([0, M])$,

$$\tilde{n} \succeq \tilde{m} \overset{\text{Def}}{\iff} \min \tilde{n}_\alpha \geq \min \tilde{m}_\alpha \text{ and } \max \tilde{n}_\alpha \geq \max \tilde{m}_\alpha \text{ for all } \alpha \in [0,1],$$

where min, max means the left or right end point of the α-cut intervals respectively.

The fuzzy strategy $\pi : \mathcal{F}_c(S) \mapsto \mathcal{F}_c(A)$ is called admissible if the α-cut set $\pi(\tilde{s})_\alpha$ of π depends only on the scalar α and the set $\tilde{s}_\alpha$, that is, it could be written as

$$\pi(\tilde{s})_\alpha = \pi(\alpha, \tilde{s}_\alpha). \tag{3.7}$$

Let Π_A be the collection of all admissible fuzzy strategies. Similarly a policy $\tilde{\pi} = (\pi_1, \pi_2, \cdots)$ is called admissible if $\pi_t \in \Pi_A$ $(t = 1, 2, \cdots)$. Our problem is to maximize $\psi(\tilde{\pi}, \tilde{s})$ over all admissible policies $\tilde{\pi}$. We define a map $\tilde{q}_\alpha : \mathcal{C}(S) \times \mathcal{C}(A) \mapsto \mathcal{C}(S)$ $(\alpha \in [0,1])$ by

$$\tilde{q}_\alpha(D \times B) := \begin{cases} \{y \in S \mid \tilde{q}(x,a,y) \geq \alpha \text{ for some } (x,a) \in D \times B\}, & \alpha > 0, \\ \mathrm{cl}\{y \in S \mid \tilde{q}(x,a,y) > 0 \text{ for some } (x,a) \in D \times B\}, & \alpha = 0, \end{cases}$$

and a map $\tilde{r}_\alpha : \mathcal{C}(S) \times \mathcal{C}(A) \mapsto \mathcal{C}([0, M])$ $(\alpha \in [0,1])$ by

$$\tilde{r}_\alpha(D \times B) := \begin{cases} \{u \in R_+ \mid \tilde{r}(x,a,u) \geq \alpha \text{ for some } (x,a) \in D \times B\}, & \alpha > 0, \\ \mathrm{cl}\{u \in R_+ \mid \tilde{r}(x,a,u) > 0 \text{ for some } (x,a) \in D \times B\}, & \alpha = 0. \end{cases}$$

Then define maps $Q^\pi_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ and $R^\pi_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}([0, M])$ $(\pi \in \Pi_A, \alpha \in [0,1])$ by

$$Q^\pi_\alpha(D) := \tilde{q}_\alpha(D \times \pi(\alpha, D)) \quad \text{and} \quad R^\pi_\alpha(D) := \tilde{r}_\alpha(D \times \pi(\alpha, D)) \quad \text{for } D \in \mathcal{C}(S).$$

For any admissible fuzzy policy $\tilde{\pi} = (\pi_1, \pi_2, \cdots)$, $Q^{\tilde{\pi}}_{t,\alpha}$ $(t \geq 1)$ is defined inductively as follows : For $D \in \mathcal{C}(S)$, $Q^{\tilde{\pi}}_{0,\alpha}(D) := D$, $Q^{\tilde{\pi}}_{1,\alpha}(D) := Q^{\pi_1}_\alpha(D)$, and

$$Q^{\tilde{\pi}}_{t+1,\alpha}(D) := Q^{(\pi_2,\pi_3,\cdots)}_{t,\alpha} Q^{\pi_1}_\alpha(D) \; (t = 1, 2, \cdots).$$

Let $V := \{v : \mathcal{C}(S) \mapsto \mathcal{C}([0, M])\}$. Define a metric d_V on V by

$$d_V(v, w) := \sup_{D \in \mathcal{C}(S)} \delta(v(D), w(D)) \quad \text{for } v, w \in V. \tag{3.8}$$

Then (V, d_V) is a complete metric space. For $v, w \in V$, we define an order

$$v \succeq_V w \overset{\text{Def}}{\iff} v(D) \succeq_{\text{ci}} w(D) \quad \text{for all } D \in \mathcal{C}(S), \tag{3.9}$$

where $\succeq_{\text{ci}}$ means that

$$[a, b] \succeq_{\text{ci}} [c, d] \overset{\text{Def}}{\iff} a \geq c \text{ and } b \geq d \quad \text{for } [a, b], [c, d] \in \mathcal{C}([0, M]).$$

Further define a map $U^\pi_\alpha : V \mapsto V$ $(\pi \in \Pi_A, \alpha \in [0,1])$ by

$$U^\pi_\alpha v(D) := R^\pi_\alpha(D) + \beta v(Q^\pi_\alpha(D)) \quad \text{for } v \in V, \; D \in \mathcal{C}(S). \tag{3.10}$$

The following results are given in [7]

Theorem 3.1.

(i) *Let* $\pi \in \Pi_A$ *and* $\alpha \in [0,1]$. *It holds that* U_α^π *is monotone, contractive and has a unique map* $v_\alpha^\pi \in V$ *such that*

$$v_\alpha^\pi = U_\alpha^\pi v_\alpha^\pi. \tag{3.11}$$

(ii) *For* $\tilde{s} \in \mathcal{F}_c(S)$ *and an admissible stationary policy* $\pi^\infty = (\pi, \pi, \pi, \cdots)$,

$$\psi(\pi^\infty, \tilde{s})_\alpha = v_\alpha^\pi(\tilde{s}_\alpha) \quad \text{for } \alpha \in [0,1]. \tag{3.12}$$

Theorem 3.2. *Let* $\check{\pi} = (\pi_1, \pi_2, \cdots)$ *be any policy.*

(i) *Suppose*

$$\psi_\alpha(\check{\pi}, D) \succeq_{ci} U_\alpha^\pi \psi_\alpha(\check{\pi}, D) \quad D \in \mathcal{C}(S), \pi \in \Pi_A, \alpha \in [0,1]. \tag{3.13}$$

Then we have

$$\psi(\check{\pi}, \tilde{s}) \succeq \psi(\check{\sigma}, \tilde{s}) \quad \tilde{s} \in \mathcal{F}_c(S), \check{\sigma} \in \Pi_A.$$

(ii) *Let* $\pi \in \Pi_A$. *Suppose*

$$U_\alpha^\pi \psi_\alpha(\check{\pi}, D) \succeq_{ci} \psi_\alpha(\check{\pi}, D) \quad \text{for all } D \in \mathcal{C}(S) \text{ and } \alpha \in [0,1]. \tag{3.14}$$

Then we have

$$\psi(\pi^\infty, \tilde{s}) \succeq \psi(\check{\pi}, \tilde{s}) \quad \text{for all } \tilde{s} \in \mathcal{F}_c(S).$$

The results like Theorem 3.2 have already appeared in the classic discounted Markov decision model and used for the policy improvement ([3, 4]). By the same idea, the above theorems would be useful for the policy improvement under the fuzzy decision model.

Next we give a fuzzy optimality equation which is used in the optimization of the decision processes. Define a map $U_\alpha : V \mapsto V$ $(\alpha > 0)$ by

$$U_\alpha v(D) := \sup_{B \in \mathcal{C}(A)} \{\tilde{r}_\alpha(D \times B) + \beta v(\tilde{q}_\alpha(D \times B))\} \quad \text{for } v \in V,\ D \in \mathcal{C}(S), \tag{3.15}$$

where sup is taken over the order $\succeq$. Then we get the following Bellman equation for each level α.

Theorem 3.3. *Let* $\alpha \in [0,1]$. U_α *is monotone, contractive and has a unique map* $v_\alpha^* \in V$ *such that*

$$v_\alpha^* = U_\alpha v_\alpha^*. \tag{3.16}$$

For $\tilde{s} \in \mathcal{F}_c(S)$, we can define

$$v^*(\tilde{s})(u) := \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{v_\alpha^*(\tilde{s}_\alpha)}(u)\} \quad u \in [0, M]. \tag{3.17}$$

Corollary 3.1.

(i) $v^* \in \mathcal{F}_c([0, M/(1-\beta)])$ *and* $v^* \succeq \psi(\check{\pi}, \tilde{s})$ *for all admissible policies* $\check{\pi}$.

(ii) *Suppose that there exists* $\pi^* \in \Pi_A$ *such that* $U_\alpha^{\pi^*} v_\alpha^* = v_\alpha^*$ *for all* $\alpha \in [0,1]$. *Then* $\pi^{*\infty}$ *is absolutely optimal, i.e.,*

$$\psi(\pi^{*\infty}, \tilde{s}) \succeq \psi(\check{\pi}, \tilde{s}) \quad \text{for all admissible policies } \check{\pi} \text{ and } \tilde{s} \in \mathcal{F}_c(S).$$

4. Fuzzy decision processes with an average reward

We specify the time average reward as a measure of the system's performance and discuss its characterization under the contractive assumption (Assumption 2.1). For a policy π^∞ with $\pi \in \Pi_A$, we define the total T-time fuzzy reward $\tilde{R}^\pi_T(\tilde{s})$ by

$$\tilde{R}^\pi_T(\tilde{s}) := \sum_{t=0}^{T-1} R(\tilde{s}_t, \pi(\tilde{s}_t)) \qquad T \geq 1, \tag{4.1}$$

where $\{\tilde{s}_t\}_{t=0}^\infty$ is given in (3.4). We estimate the increasing amount of fuzzy reward per unit time. For $\pi \in \Pi_A$, $K > 0$ and $\alpha \in [0,1]$, we define

$$G^\pi_{K,\alpha} := \left\{ r \in \mathbf{R}_+ \;\middle|\; \begin{array}{l} \text{there exists } \{z_T\}_{T=0}^\infty \text{ such that} \\ z_T \in \tilde{R}^\pi_T(\tilde{s})_\alpha \text{ and } |z_T - rT| \leq K \text{ for all } T \geq 0 \end{array} \right\}. \tag{4.2}$$

Then the following fuzzy number could be defined.

$$\tilde{g}^\pi(\tilde{s})(r) := \sup_{\alpha\in[0,1]} \{\alpha \wedge 1_{G^\pi_{K,\alpha}}(r)\} \quad r \in [0, M] \quad \text{for } \pi \in \Pi_A,\ \tilde{s} \in \mathcal{F}(S). \tag{4.3}$$

Theorem 4.1. ([8]) *Under adequate conditions(contraction and continuity), it holds that*

$$\lim_{T\to\infty} \tilde{R}^\pi_T(\tilde{s})/T = \tilde{g}^\pi(\tilde{s}).$$

Theorem 4.2. ([9]) *Under the same conditions in Theorem 4.1,*

(i) *There exists bounded functions $\underline{h}^\pi_\alpha$ and $\overline{h}^\pi_\alpha$ on $\mathcal{C}(S)$ and for $D \in \mathcal{C}(S)$, the followings hold:*

$$\underline{h}^\pi_\alpha(D) + \min \tilde{g}^\pi(\tilde{s})_\alpha = \min R^\pi_\alpha(D) + \underline{h}^\pi_\alpha(Q^\pi_\alpha(D)), \tag{4.5}$$

$$\overline{h}^\pi_\alpha(D) + \max \tilde{g}^\pi(\tilde{s})_\alpha = \max R^\pi_\alpha(D) + \overline{h}^\pi_\alpha(Q^\pi_\alpha(D)). \tag{4.6}$$

(ii) *Conversely, if there exist bounded functions $\underline{h}^\pi_\alpha$ and $\overline{h}^\pi_\alpha$ on $\mathcal{C}(S)$ and constants $\underline{K}^\pi_\alpha$ and $\overline{K}^\pi_\alpha$ satisfying the following equations :*

$$\underline{h}^\pi_\alpha(D) + \underline{K}^\pi_\alpha = \min R^\pi_\alpha(D) + \underline{h}^\pi_\alpha(Q^\pi_\alpha(D)), \tag{4.7}$$

$$\overline{h}^\pi_\alpha(D) + \overline{K}^\pi_\alpha = \max R^\pi_\alpha(D) + \overline{h}^\pi_\alpha(Q^\pi_\alpha(D)) \tag{4.8}$$

for all $D \in \mathcal{C}(S)$, then $(\tilde{g}^\pi(\tilde{s}))_\alpha = [\underline{K}^\pi_\alpha, \overline{K}^\pi_\alpha]$.

5. Conclusion

Under the the stochastic dynamics system the decision processes is known as Markov decision processes. The fundamental theory of Markov decision process are established fruitfully and they are applied so many fields of practical applications. This theory is a direct expansion from the discrete system of the deterministic motion law. As is discussed in this paper, the fuzzy decision process(abr. FDP) is formulated as the extension of the deterministic decision process and this fuzzyfication should be expected to apply so many fields. Alternatively we have given the formulation and the analysis of FDP via using the manipulation or operation for fuzzy sets. The important notion in the convergence and the partial order owes much to the discussion of the class of the closed convex sets which are reduced by the α-cuts of the convex fuzzy sets. This method is rather simple in the mathematical treatment and hence it is efficient for the much more complex systems such as a sequential fuzzy game. However we must solve the optimization problem for which the preference has a partial order. So it faces a similar difficulty as in the multi-objective problem. To overcome these difficulty, a solution method of Linear Programming should be included considering the application to clarify the complex structure.

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References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* 17 (1970) 141-164.

[2] D.P.Bertsekas and S.E.Shreve, *Stochastic Optimal Control : the Discrete Time Case* (Academic Press, New York, 1978).

[3] D.Blackwell, Discrete dynamic programming, *AMS* 33 (1962) 719-726.

[4] D.Blackwell, Discounted Dynamic Programming, *Ann. Math. Stat.*, 36 (1965) 226-235.

[5] D.Dubois and H.Prade, Systems of linear fuzzy constraints, *Fuzzy Sets and Systems* 3 (1980) 37-48.

[6] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* 51 (1992) 83-88.

[7] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, Markov-type fuzzy decision processes with a discounted reward on a closed interval, *Technical Reports of Mathematical Sciences, Chiba University,* No.10, Vol.4 (1992).

[8] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, The time average reward with some dynamic fuzzy systems, *Technical Reports of Mathematical Sciences, Chiba University,* No.10, Vol.5 (1992).

[9] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, Fuzzy decision processes with an average reward criterion, *Technical Reports of Mathematical Sciences, Chiba University,* No.4, Vol.11 (1995).

[10] K.Kuratowski, Topology I, *Academic Press*, New York, 1966.

[11] S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* 33 (1989) 123-126.

[12] V.Novàk, *Fuzzy Sets and Their Applications* (Adam Hilder, Bristol-Boston, 1989).

[13] S.M.Ross, *Applied probability models with optimization applications* (Holden-Say, San Francisco, 1970).

[14] M.Yasuda, A linear structure and convexity for relations in dynamic fuzzy systems, *Computers Math. Applic.* 27 (1994) 23-27.

[15] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The contractive case, *Fuzzy Sets and Systems* 60 (1993) 283-294.

[16] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure: The unbounded case, *Fuzzy Sets and Systems* 66 (1994) 83-95.

[17] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A limit theorem in dynamic fuzzy systems with a monotone property, *Technical Reports of Mathematical Sciences, Chiba University,* No.5, Vol.11 (1995).

[18] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems.* 66 (1994) 39-57.

[19] Y.Yoshida, The recurrence of dynamic fuzzy systems, *RIFIS Technical Report*, No. 88 (1994), Research Institute of Fundamental Information Science Kyushu University 33 Fukuoka 812, Japan.

[20] Y.Yoshida, Optimization on some dynamic Fuzzy systems, *Plenary talk in the Statistical division at the annual meeting of Mathematical Society of Japan*, 1994.

[21] L.A.Zadeh, Fuzzy sets, *Inform. and Control* 8 (1965) 338-353.