The "Inequity" Spectra of a Lifetime Distribution

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The Gini Index, and the Lorenz Curve, are two commonly used metrics in econometrics [cf. Atkinson (1970), Rothschild and Stiglitz (1973)]. In reliability theory, it is the mean time to failure, and the failure rate; see, for example, Barlow and Proschan (1964). However, most individuals do not conceptualize in terms of metrics such as these. People are unable to relate to them because they, being transforms of the observed data, are not observable, and often not actionable. Philosophically, they violate the positivist ideas of Mach and Einstein. In actuality, individuals are attuned to thinking in terms of more intuitive notions, like the disparity between an observed value of a variable and its target. The standard deviation is also a transformation of the data, namely, the average of all the squared deviations from a sample mean. Furthermore, the mean may not coincide with an economist’s ideal value, or an engineer’s target (design) value.

I introduce here a new notion, a notion which comes closer to the Mach-Einstein ideal of positivism. This notion, which I have labeled "inequity", is possibly novel to reliability, and to survival analysis. In this presentation, my aim is to argue that there must exist a notion of inequity, and to discuss methods for quantifying inequity. As a guidepost, to doing the above, I lean on the theory of subjective probability, and decision making under uncertainty, as enunciated by von Neumann and Morgenstern (1947). In a theory, like this, one starts with the undefined primitive of one’s ability to judge that "an event, say A is more liable to occur than an event, say B"; one then imposes on this primitive, several axioms of rational behavior. Doing so leads one to mathematically deduce the metric of (subjective) probability, as an approach to quantifying uncertainty [cf. De Groot (1970)]. Similarly, with the notion of utility wherein a primitive is one’s ability to choose between actions.

Here, my primitive is an intuitive sense about the extent to which all the observed values of a variable depart from a target, and an ability to declare which of the two collections of variables come closer to their targets. The axiom part pertains to "equity", which occurs when all the probability mass of a random variable is concentrated at a target value. Then, there exists a spectrum of inequalities which encapsulate a quantified inequity. A setup such as this comes the closest in spirit to a linkage between games, inequity, and reliability.

When it comes to the matter of quantified measures of inequity, there is precedence in econometrics, mentioned above, namely, the Lorenz Curve, and the Gini Index. The Lorenz Curve and its dual the Leimkuhler Curve inform economists and social scientists the degree of social imbalance or unfairness in a community. The Lorenz Curve spawns the Gini Index which has turned out to be one of the most popular measures of wealth concentration in a society. It has been used by welfare economists for ranking countries in terms of their social imbalance, and also for setting taxation policies. Looking to the future, the Gini Index may facilitate a deeper perspective on the nature of trade imbalances, a current topic of much debate and discussions. This development enables one to place the Lorenz Curve, and hence the Gini Index, in a more general and broader perspective. We refer to the target value as an anchor point, and consider deviations from anchor points. Examples of anchor points are the mean, the mode, the median, or any percentile of a probability distribution. Indeed, an anchor point need not even belong to the support of a distribution. In life testing for reliability and survival analysis, the above inequity measures can inform one about the "riskiness" of engineered systems. They can also be used for evaluating
competing designs, or the efficacy of drugs and medical procedures. For example, the Gini Index can be used as a measure which informs an engineer about the extent to which the service lives of a group of items deviate from their design value.

The inequality measures mentioned above suffer from a limitation because their construction is based on either shortfalls or excesses, from a target, be they incomes or lifetimes, but not on a conglomeration of the two. A conglomeration can provide a panoramic perspective of an underlying inequity, and enable deeper insights about the nature of an imbalance.

The proposed new measure is a concatenation of inequalities in two directions, shortfall as well as exceedance, with respect to the anchor point. With the mean as an anchor point, the proposed measure enables a decomposition of the Gini Index; see theorem below. The decomposition provides an interpretation of the Gini Index which is more constructive than that provided by the Lorenz Curve; it also enables a generalization of the Gini Index. Different anchor points lead to different spectra. Each spectrum yields its own measure of concentration, and this spawns the generalization mentioned above. The generalization has a merit. It enables a country, the freedom to choose that anchor point which best reflects what it envisions an egalitarian society to be. The Gini Index is based on the mean as an anchor point. There is a probabilistic import of the proposed spectrum. It enables one to characterize distributions in terms of their hazard functions, such as those with increasing, decreasing, or non-monotonic failure rate functions that characterize wear, degradation, and ageing.

To summarize, merits of the proposed spectrum are: a deeper insight into the nature of imbalance, be it economic or lifetimes, a decomposition and generalization of the Gini Index, the freedom it offers a society for envisioning its view of what it means to be egalitarian, and another approach for characterizing and classifying lifetime and income distributions.

**Theorem:** The Gini Index of a distribution $F$ with mean $\mu$, say $G_F$ can be decomposed as

$$G_F = \frac{2}{\mu} \left[ \int_0^{1-F(\mu)} P_\mu(u) du + \int_{1-F(\mu)}^{+\infty} D_\mu(u) du \right].$$

Here $D_\mu(p) = \mu(1-p) - \mu L_F(1-p)$, and $P_\mu(p) = \mu[1 - L_F(1-p) - p]$. $L_F(p)$ is the Lorenz curve of $p$.

**References**


