Investment Timing and Capacity Decisions with Time-to-Build in a Duopoly Market

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1 Introduction

We study firms’ optimal decisions of investment timing and capacity with time-to-build and competition. We show that a firm dominated in terms of investment lags can invest earlier than the dominant one in a market equilibrium, whereas the latter always invests earlier than the former in socially optimal strategies.

2 Setup

Type A and B firms produce homogenous goods. The price at time $t$ is $P(t) = X(t)(1 - \eta Q(t))$ where $Q(t)$ is total market output, $\eta > 0$ is a constant, and $X(t)$ is a demand shock. The demand shock follows $dX(t) = \mu X(t)dt + \sigma X(t)dw(t)$ where $\mu$ and $\sigma$ are constants and $(W_t)_{t \geq 0}$ is a standard Brownian motion. The risk-free rate is given by a constant $r(> \mu)$ and the investment incurs costs $\delta$ per unit. Type $i$ firm’s time-to-build $\tau_i$ follows $\exp(\Lambda_i)$ for $i \in \{A, B\}$.

3 Main results

We designate the firms’ roles based on their investment timing. For now, we assume type $i$ and $j$ firms are a leader and a follower, respectively, for $i, j \in \{A, B\}$ and $i \neq j$. Given type $i$ leader’s product in the market, type $j$ follower’s value is

$$V_{F1}^j(X, Q^j_L) = \max_{T^{F1}_j \geq 0, Q^{F1}_j \geq 0} \mathbb{E}\left[ \int_{T^{F1}_j}^{\infty} e^{-r t} Q^j_{F1} X(t) \right] \times (1 - \eta(Q^j_L + Q^j_{F1})) dt - e^{-r \tau_j} X(0) = X. \]

where $T^{F1}_j := T^{F1}_j + \tau_j$. Type $j$ follower’s investment timing can be described as $T^{F1}_j := \inf\{t \geq T^{F1}_j | X(t) \geq X^{F1}_j\}$ where $T^{F1}_j$ denotes type $i$ leader’s manufacturing timing, and we have the following result:

**Proposition 1** Provided that type $i$ leader’s product has entered the market, type $j$ follower’s optimal capacity corresponding to the demand shock $X$ and the leader’s capacity $Q^j_L$ is

$$Q^{\ast j}_{F1}(X, Q^j_L) = \frac{1}{2\eta} \left( 1 - \eta Q^j_L - \frac{\delta(r - \mu)(r + \lambda_j - \mu)}{\lambda_j X} \right).$$

The value function of the follower after the completion of the leader’s project is

$$V_{F1}^j(X, Q^j_L) = \begin{cases} A_{F1}^j(Q^j_L) X^j, & \text{if } X < X^{\ast j}_{F1}(Q^j_L), \\ \frac{d}{\lambda_j} \frac{(r - \mu)(r + \lambda_j - \mu)}{\lambda_j X}, & \text{if } X \geq X^{\ast j}_{F1}(Q^j_L), \end{cases}$$

where $A_{F1}^j(Q^j_L) = (1 - \eta Q^j_L) (\beta - 1)/(\eta \delta - (\beta + 1)(r - \mu)^2 + \sqrt{1/2 - \mu/\sigma^2}) > 1$. The optimal investment threshold and the corresponding optimal capacity are given by $X^{\ast j}_{F1}(Q^j_L) = (\beta + 1) \delta(r - \mu)(r + \lambda_j - \mu)/((\beta - 1)(1 - \eta Q^j_L) \lambda_j)$ and $Q^{\ast j}_{F1}(Q^j_L) := Q^{\ast j}_{F1}(X^{\ast j}_{F1}(Q^j_L), Q^j_L) = (1 - \eta Q^j_L)/((\beta + 1) \eta)$.

Suppose type $i$ leader invested in $Q^j_L$ but its product has not been produced yet. The follower’s value is

$$V_{F0}^j(X, Q^j_L) = \max_{T^{F0}_j \geq 0, Q^{F0}_j \geq 0} \mathbb{E}\left[ \int_{T^{F0}_j}^{\infty} e^{-r t} Q^j_{F0} X(t) \right] \times (1 - \eta(Q^j_L + Q^j_{F0})) dt - e^{-r T^{F0}_j} \delta Q^j_{F0} + 1 \{T^{F0}_j < T^{F1}_j\} \int_{T^{F0}_j}^{\infty} e^{-r t} Q^j_{F0} X(t) (1 - \eta Q^j_{F0}) dt + 1 \{T^{F0}_j \geq T^{F1}_j\} \int_{T^{F1}_j}^{\infty} e^{-r t} Q^j_{F0} X(t) \left( (1 - \eta Q^j_{F0} + Q^j_{F0}) dt - e^{-r T^{F0}_j} \delta Q^j_{F0} \right)
+ 1 \{T^{Fj} \leq T^{F0}_j\} e^{-r T^{Fj}_j} V_{F1}^j(X^{F0}, X(0) = X),$$

where $T^{F0}_j := T^{F0}_j + \tau_j$. The follower’s investment timing can be written as $T^{F0}_j := \inf\{t \in (0, T^{F0}_j] | X(t) \geq X^{F0}_j\}$, and we can derive the following result:

**Proposition 2** Provided that type $i$ leader’s product has not entered the market yet, type $j$ follower’s optimal capacity corresponding to the demand shock $X$
and the leader’s capacity $Q_L^*$ is
\[
Q_{F0}^*(X, Q_L) = \frac{1}{2\eta} \left[ -\frac{\delta(r - \mu)(r + \lambda_j - \mu)}{\lambda_j X} + \frac{(1 - \eta Q_L)(\lambda + (r + \lambda_j - \mu))}{r + \lambda_j + \lambda_j - \mu} \right].
\]
(1)

The value function of the follower before the completion of the leader’s project is
\[
V_{F0}^*(X, Q_L) = \begin{cases} A_{F1}^*(Q_L^*)X^\beta + A_{F0}^*(Q_L^*)X^{\beta_0}, & \text{if } X < X_{F0}^*(Q_L^*), \\ B_{F0}^*(X, Q_L), & \text{if } X \geq X_{F0}^*(Q_L^*), \end{cases}
\]
where $A_{F0}^*(Q_L^*)$ and $B_{F0}^*(X, Q_L)$ can be found in the original paper and $\beta_i$ denotes $\beta$ with $r + \lambda_i$ instead of $r$. The optimal investment threshold and capacity, $X_{F0}^*(Q_L^*)$ and $Q_{F0}^*(Q_L^*) := Q_{F0}^*(X_{F0}^*(Q_L^*), Q_L^*)$, are implicitly derived from (1) and
\[
\delta(1 - \eta Q_L) (\beta - \beta) \left( \frac{X_{F0}^*(Q_L^*)}{X_{F1}^*(Q_L^*)} \right)^{\beta} + \beta_0 Q_{F0}^*(Q_L^*)
\]
\[
= \frac{(\beta - 1) Q_{F0}^*(Q_L^*)}{(r - \mu)(r + \lambda_j + \lambda_j - \mu)} \left[ 1 - \eta Q_L^* \right] \left( \frac{\lambda}{r + \lambda_j - \mu} \right) + \frac{(1 - \eta Q_L^*)}{r + \lambda_j - \mu} \lambda.
\]

The leader takes the follower’s investment decisions into account, and its value is evaluated as follows:
\[
V_L^*(X) = \max_{T_L \geq 0, Q_L^* \geq 0} \mathbb{E} \left\{ \int_{T_L^0}^{T_L^0} e^{-rT_L} X(t) dt + \int_{T_L^0}^{T_L} e^{-rT_L} X(t) (1 - \eta(Q_L + Q_{F0}^*)) dt \right\}
\]
\[
+ \int_{T_L^0}^{T_L} e^{-rT_L} X(t) dt + e^{-rT_L} X(t) (1 - \eta(Q_L + Q_{F0}^*)) dt \right\}
\]
\[
+ \int_{T_L^0}^{T_L} e^{-rT_L} X(t) (1 - \eta(Q_L + Q_{F0}^*)) dt + \delta T_L Q_{L}^*.
\]

The leader’s investment timing is $T_L := \inf \{ t \in (0, T_{F0}] | X(t) \geq X_L^* \}$, and we have the following:

**Proposition 3** Type $i$ leader’s optimal capacity corresponding to demand shock $X$ is
\[
Q_{L}^*(X) = \arg\max_{Q_L^* \geq 0} A_L^*(X, Q_L^*),
\]
where $A_L^*(X, Q_L^*)$ can be found in the original paper. The value function of the leader is
\[
V_L^*(X) = \begin{cases} A_L^*(X, Q_L^*) \left( \frac{X}{X_L^*} \right)^{\beta}, & \text{if } X \leq X_L^*, \\ A_L^*(X, Q_L^*(X)), & \text{if } X \geq X_L^*, \end{cases}
\]
where the leader’s optimal investment threshold and capacity are $X_L^* = \arg\max A_L^*(X_L^*, Q_L^*(X_L^*))$ and $Q_L^* := Q_{L}^*(X_L^*)$, respectively.

If we characterize type $i$ and $j$ firms’ roles as a leader and a follower, respectively, and their investment strategies by a tuple $S_{ij} = (i, j, X_L^*, X_{F0}^*, X_{F1}^*, Q_L^*, Q_{F0}^*, Q_{F1}^*)$, we can immediately obtain the following result:

**Corollary 1** Given the exogenous roles of type $i$ and $j$ firms as a leader and a follower, respectively, the optimal strategy is
\[
S_{ij} = (i, j, X_L^*, X_{F0}^*, X_{F1}^*, Q_L^*, Q_{F0}^*, Q_{F1}^*).
\]

Now we endogenize the firms’ roles. If type $i$ firm invests given type $j$’s investment, its value as a follower is $V_{F0}^*(X, Q_L^*)$. If type $i$ firm invests given demand shock $X$ to preempt the market, its value as a leader is
\[
V_L^*(X) = \begin{cases} A_L^*(X, Q_L^*(X)), & \text{if } X < X_{F0}^*(Q_L^*(X)), \\ Q_L^*(X)^{1/\delta} (\mu + 1) X^{\frac{1 - \eta Q_L^*}{\eta Q_L^* + \lambda}} \left( 1 - \eta Q_L^* \right) \left[ 1 + \frac{(1 - \eta Q_L^*)}{\eta Q_L^* + \lambda} \right] - \delta Q_L^* \left( X, \right), & \text{if } X \geq X_{F0}^*(Q_L^*(X)). \end{cases}
\]

A firm only has an incentive to preempt the market when its value as a leader exceeds that as a follower, and we can conjecture $X_p^* := \inf \{ X > 0 | V_L^*(X, Q_L^*) \}$ for $i \in \{A, B\}$. Given the assumption of $X_p^* \cup X_p^* \neq \emptyset$ for $i \neq j$, we have the following:

**Proposition 4** The level of demand shock at which type $i$ firm is indifferent between being a leader and being a follower is derived from
\[
V_L^*(X_p^*) = V_{F0}^*(X_p^*, Q_{F0}^*),
\]
where $Q_{F0}^* := Q_{F0}^*(X_p^*)$ for $i, j \in \{A, B\}$ and $i \neq j$.

(1) If $X_p^* \neq \emptyset$ for $i \in \{A, B\}$ and $X_p^* \leq X_p^*$ for $i \neq j$, the optimal strategy is
\[
S^* = (i, j, X_p^*, X_{F0}^*(Q_{F0}^*(X_p^*)), X_{F1}^*(Q_{F0}^*(X_p^*)), X_p^*, Q_{F0}^*(Q_{F0}^*(X_p^*)), Q_{F1}^*(Q_{F0}^*(X_p^*)�),
\]
where $Q_{F0}^* := Q_{F0}^*(X_p^*)$.

(2) If $X_p^* \neq \emptyset$ and $X_p^* > X_p^*$ for $i \neq j$, the optimal strategy is $S^* = S_{ij}$.

**References**