A new random projection method for Linear Programming

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1. Introduction

In this paper we propose a new random projection method that allows to reduce the number of inequalities of a Linear Program (LP). More precisely, we randomly aggregate the constraints of a LP into a new one with much fewer constraints, while preserving, approximately, the optimal value of the LP. This section extends the work in [3], where the authors considered an LP with equality constraints and proved that we can build a new LP, whose optimal value approximate the original one, with much fewer constraints by randomly aggregating them using a matrix of independent and identically distributed (iid) random variables. The main tool, of the above paper, is the Johnson-Lindenstrauss Lemma [1], which states that a set of high-dimensional points can be projected to a much lower dimensional one, while preserving, approximately, the Euclidean distance between these points. The extension, to the inequality case, proposed in this paper is non-trivial as we need to consider random matrix with sub-exponential (and not sub-gaussian) entries, which forbid the use of the Johnson-Lindenstrauss Lemma directly.

In the whole paper we consider a random projection matrix $T \in \mathbb{R}^{k \times m}$ such that $T = \frac{1}{\sqrt{k}} G$ where $G$ is a random matrix whose entries $g_{ij}$ are independent random variables normally distributed.

Let us consider the following bounded LP:

$$\min_{x \in \mathbb{R}^n} \begin{cases} c^\top x \\ Ax \geq b \end{cases} \quad \text{subject to} \quad P \quad (1)$$

where $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ and let us denote by $v(P)$ the value of the objective at an optimal solution (that we assume finite). Given a random $k \times m$ matrix $T$ we define the following “projected” LP:

$$\min_{x \in \mathbb{R}^n} \begin{cases} c^\top x \\ Ax \geq b \\ TD(Ax - b)T^\top \geq 0 \end{cases} \quad \text{subject to} \quad P_t \quad (2)$$

where $D(\cdot) : \mathbb{R}^p \mapsto \mathbb{R}^{p \times p}$ maps a vector to a diagonal matrix and where $d(\cdot) : \mathbb{R}^{p \times p} \mapsto \mathbb{R}^p$ returns the diagonal of a matrix.

Notice that for any feasible solution $x$ of $P$ the matrix $TD(Ax - b)T^\top$ is PSD hence its diagonal is a non-negative vector of $\mathbb{R}^k$ which implies that any feasible solution of $P$ is also a feasible solution of $P_t$, hence we deduce that $v(P_t) \leq v(P)$.

Hence the aim of this paper is to prove that we can find $\delta > 0$ such that

$$v(P) - \delta \leq v(P_t) \leq v(P)$$

holds.

Let $S \in \mathbb{R}^{k \times m}$ be the random matrix whose entries are the square of the entries of $T$, i.e. $S_{ij} = T_{ij}^2$. We can prove that (2) is equivalent to

$$\min_{x \in \mathbb{R}^n} \begin{cases} c^\top x \\ SdAx \geq \geq Sb \end{cases} \quad \text{subject to} \quad P_t \quad (3)$$

2. Random projections

In this section we recall some facts about concentration inequalities.

定義 1. Let $X$ be a zero mean random variable such that there exists $K > 0$ such that for all $t > 0$:

$$P(|X| > t) \leq 2e^{-\frac{t^2}{2K^2}} \quad (4)$$

Then $X$ is said to be sub-gaussian. The sub-gaussian norm, $\|X\|_{2\varepsilon}$, of $X$ is defined to be the smallest $K$ satisfying (4).

定義 2. Let $X$ be a zero mean random variable such that there exists $K > 0$ such that for all $t > 0$:

$$P(|X| > t) \leq 2e^{-\frac{t^2}{2K^2}} \quad (5)$$
Then $X$ is said to be sub-exponential. The sub-exponential norm, $\|X\|_{\psi_1}$, of $X$ is defined to be the smallest $K$ satisfying (5).

Sub-gaussian and sub-exponential random variables are closely related, as we can see any sub-gaussian random variable is also sub-exponential, furthermore, it turns out that the product of two sub-gaussian random variables is sub-exponential.

We now recall the famous Johnson-Lindenstrauss lemma.

**Lemma 2.1** (Johnson-Lindentrauss Lemma [2]). Let $X$ be a set of $n$ points in $\mathbb{R}^m$ and let $G$ be a $k \times m$ random matrix whose rows are independent, mean zero, sub-gaussian and isotropic, then with probability $1 - 2 \exp(-C_0k\varepsilon^2)$, we have that for all $x_i, x_j \in X$

\[
(1-\varepsilon)\|x_i - x_j\|_2 \leq \frac{1}{\sqrt{k}}\|Gx_i - Gx_j\|_2 \leq (1+\varepsilon)\|x_i - x_j\|_2
\]

(6)

where $C_0$ is an universal constant.

Notice that this lemma implies that

\[
k = O\left(\frac{\log(n)}{\varepsilon^2}\right)
\]

3. **Lower bound for the value of $v(P_T)$**

Let us consider the duals, $D$ of (1) and $D_T$ of (2):

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^\top y \\
A^\top y & = c \\
y & \geq 0
\end{align*} \quad \begin{cases} 
D
\end{cases}
\]

(7)

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^\top d(T^\top D(y)T) \\
A^\top d(T^\top D(y)T) & = c \\
y & \geq 0
\end{align*} \quad \begin{cases} 
D_T
\end{cases}
\]

(8)

Let $y^* \in \mathbb{R}^m_+$ be an optimal solution of (7), we consider the following “approximated” projected solution:

\[
y_T = d(TD(y^*)T^\top) \in \mathbb{R}^k
\]

Notice that $y_T \geq 0$ (as $TD(y^*)T^\top$ is PSD by definition of $y^*$). We will now prove that $y_T$ is almost feasible for (8).

Let us consider the modified dual problem:

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^\top d(T^\top D(y)T) \\
A^\top d(T^\top D(y)T) & = c + A^\top (d(T^\top D(y_T)T) - y^*) \\
y & \geq 0
\end{align*}
\]

(9)

Obviously $y_T$ is a feasible solution of the above LP. In order to obtain a lower bound for $v(P_T)$ we prove the following:

- the value $b^\top d(T^\top D(y_T)T)$ can be written as $b^\top y^* + O(\varepsilon)$
- the vector $A^\top (d(T^\top D(y_T)T) - y^*)$ can be written as $O(\varepsilon)1$ (where $1$ is the all-ones vector)

Then considering the dual of the modified dual problem we obtain a lower bound $v(P) - \delta$ for $v(P_T)$ where $\delta = O(\varepsilon)$.

参考文献

