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Min-Max Formulas for Separable Discrete Convex Minimization on Box-TDI Polyhedra

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1. Introduction

In discrete convex analysis [6], Fenchel-type min-max formulas are discussed for L- and M-convex functions and convex-cost integral flows. The following is an example of such min-max formula for minimization of square-sum of components over an M-convex set (the set of integer points of an integral base-polyhedron).

Theorem 1 ([4])**.** *Let $B \subseteq \mathbf{R}^n$ be an integral base-polyhedron and p be the associated integer-valued supermodular function. Then*

$$\min\{\sum_{1 \le i \le n} z_i^2 : z \in B \cap \mathbf{Z}^n\}$$
$$= \max\{\hat{p}(w) - \sum_{1 \le i \le n} \left\lfloor \frac{w_i}{2} \right\rfloor \left\lceil \frac{w_i}{2} \right\rceil : \ w \in \mathbf{Z}^n\},$$

where $\hat{p}$ is the linear (or Lovász) extension of p.

The objective of this paper is show that the discrete Fenchel-type min-max formula holds for integer-valued separable convex functions defined on the set of integral elements of a box-TDI polyhedron R. Details are given in [5].

An important special case is where R is described by a totally unimodular matrix. This includes the case of L-convex or $\mathrm{L}^\natural$-convex sets. It can be proved that $\mathrm{L}_2^\natural$-convex (in particular, L_2-convex) sets are also discrete box-TDI sets. Another special case is the one of integral submodular flows, in particular, M_2-convex and $\mathrm{M}_2^\natural$-convex sets.

2. TDI and discrete convex function

A (rational) linear system $Qx \ge p$ is called *totally dual integral* (or *TDI*) [1] if the maximum in the LP duality

$$\min\{cx : Qx \ge p\} \ = \ \max\{yp : y \ge 0, yQ = c\}$$

has an integral optimal solution y for every integral vector c for which the maximum is finite. The system $Qx \ge p$ is called *box-totally dual integral* (or *box-TDI*) if the system $[Qx \ge p, f \le x \le g]$ is TDI for every choice of rational bounding functions $f \le g$. A polyhedron is called a *box-TDI polyhedron* if it can be described by a box-TDI system [1, 3]. See also [2, 7].

A function $\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ is called *discrete convex* if $\varphi(k-1) + \varphi(k+1) \ge 2\varphi(k)$ for each k with $\varphi(k) < +\infty$. A function $\Phi : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ is called a *separable discrete convex function* if it is represented as

$$\Phi(z) = \sum_{1 \le i \le n} \varphi_i(z_i) \qquad (1)$$

with discrete convex functions $\varphi_1, \ldots, \varphi_n$. The *discrete conjugate* $\varphi^\bullet$ of a function $\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ is defined, for integers ℓ, by

$$\varphi^\bullet(\ell) = \max\{k\ell - \varphi(k) : k \in \mathbf{Z}\}.$$

The discrete conjugate $\Phi^\bullet$ of Φ is given, for $w \in \mathbf{Z}^n$, by

$$\Phi^\bullet(w) = \max\{wz - \Phi(z) : z \in \mathbf{Z}^n\} = \sum_{1 \le i \le n} \varphi_i^\bullet(w_i),$$

where wz denotes the inner product of w and z.

3. Main results

Let R be an integral box-TDI polyhedron in $\mathbf{R}^n$. We assume that Φ is an integer-valued separable discrete convex function, finite-valued and bounded from below on $R \cap \mathbf{Z}^n$.

The following is a Fenchel-type duality theorem using function $\mu_R(w) = \min\{wx : x \in R\}$, which coincides with $\hat{p}(w)$ when R is an integral base-polyhedron.

Theorem 2. *Let R be an integral box-TDI polyhedron in $\mathbf{R}^n$. Then*

$$\min\{\Phi(z) : z \in R \cap \mathbf{Z}^n\}$$
$$= \max\{\mu_R(w) - \Phi^\bullet(w) : w \in \mathbf{Z}^n\}.$$

The second theorem makes explicit reference to a description of R.

Theorem 3. *Let Q be an integral $m \times n$ matrix and p an integral vector. Assume that $Qx \geq p$ is box-TDI and let $R = \{x : Qx \geq p\}$. Then*

$$\min\{\Phi(z) : z \in R \cap \mathbf{Z}^n\}$$
$$= \max\{yp - \Phi^\bullet(yQ) : y \geq 0, \ \text{integer-valued}\},$$

where the optimal $y \in \mathbf{Z}^m$ can be chosen to have at most $2n$ positive components.

4. Special cases

Theorem 4 (Square-sum on box-TDI set). *Let $R = \{x : Qx \geq p\}$ be a box-TDI polyhedron.*

$$\min\{\sum_{1\leq i\leq n} z_i^2 : z \in R \cap \mathbf{Z}^n\}$$
$$= \max\{yp - \sum_{1\leq i\leq n} \left\lfloor \frac{w_i}{2} \right\rfloor \left\lceil \frac{w_i}{2} \right\rceil : \ w = yQ, y \in \mathbf{Z}^m_+\},$$

where $z^* \in R \cap \mathbf{Z}^n$ is a minimizer if and only if there exists an integer vector $y^* \geq 0$ such that $y^*(Qz^* - p) = 0$ and $\lfloor y^*Q/2 \rfloor \leq z^* \leq \lceil y^*Q/2 \rceil$. More generally, for positive $c \in \mathbf{Z}^n$,

$$\min\{\sum_{1\leq i\leq n} c_i z_i^2 : z \in R \cap \mathbf{Z}^n\}$$
$$= \max\{yp - \sum_{1\leq i\leq n} \left\lfloor \frac{w_i + c_i}{2c_i} \right\rfloor \left(w_i - c_i \left\lfloor \frac{w_i + c_i}{2c_i} \right\rfloor \right),$$
$$w = yQ : \ y \geq 0 \ \text{integral}\},$$

where $z^* \in R \cap \mathbf{Z}^n$ is a minimizer if and only if there exists an integer vector $y^* \geq 0$ such that $y^*(Qz^* - p) = 0$ and $\lceil (y^*Q - c)/(2c) \rceil \leq z^* \leq \lfloor (y^*Q + c)/(2c) \rfloor$.

Theorem 5 (Square-sum on M_2-convex set [4]). *Let B_1 and B_2 be integral base-polyhedra with associated supermodular functions p_1 and p_2, where $B_1 \cap B_2 \neq \emptyset$ is assumed. Then*

$$\min\{\sum_{1\leq i\leq n} z_i^2 : z \in B_1 \cap B_2 \cap \mathbf{Z}^n\}$$
$$= \max_{w,u \in \mathbf{Z}^n} \{\hat{p}_1(w) + \hat{p}_2(u) - \sum_{1\leq i\leq n} \left\lfloor \frac{w_i + u_i}{2} \right\rfloor \left\lceil \frac{w_i + u_i}{2} \right\rceil\},$$

where $\hat{p}_k$ is the linear (Lovász) extension of p_k.

Network flow: Let $D = (V, A)$ be a digraph and $m \in \mathbf{Z}^V$ be a vector with zero component-sum. An *m-flow* means a flow (a vector on A) that meets the supply-demand requirement represented by m. For a potential function π on V, the potential difference is denoted by $\Delta_\pi(uv) = \pi(v) - \pi(u)$ for $uv \in A$.

Theorem 6. *The minimum square-sum of an integral m-flow is equal to*

$$\max\{m\pi - \left\lfloor \frac{\Delta_\pi}{2} \right\rfloor \left\lceil \frac{\Delta_\pi}{2} \right\rceil : \pi : V \to \mathbf{Z}_+\}.$$

The minimum square-sum of a non-negative integral m-flow is equal to

$$\max_{\pi:V\to\mathbf{Z}_+} \{m\pi - \left\lfloor \frac{\max(\Delta_\pi, 0)}{2} \right\rfloor \left\lceil \frac{\max(\Delta_\pi, 0)}{2} \right\rceil\}.$$

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