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STOCHASTIC MARKSMANSHIP CONTEST GAMES WITH RANDOM TERMINATION — SURVEY AND APPLICATIONS

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Abstract The marksmanship contest game with random termination is a fundamental two-person game of timing under an uncertain environment. In this paper, we formulate a unified marksmanship contest game with random termination and show that it can be further classified into five different games. We derive the optimal solutions of these games of timing, which are categorized into the Nash equilibrium strategy and the Stackelberg strategy. Our results involve the well-known results by Teraoka (1983) and Baston and Garnaev (1995) as special cases and at the same time give new insights for different types of game.

Keywords: Game theory, silent-type game, noisy-type game, marksmanship contest game with random termination, Nash equilibrium strategy, Stackelberg strategy

1. Introduction

The game of timing is a fundamental tool to determine the best timing for players with conflict of interests. The duel game, which is the representative example of two-person game of timing, is formulated as the problem in which two rival players determine the best timings of fire to win. In general, duel games can be categorized into two types: zero-sum duel game and nonzero-sum duel game. The zero-sum duel game is formulated to describe the situation where the sum of payoffs for two players is always constant. On the other hand, the nonzero-sum duel game is modeled as the game where the sum of rewards for both players is not constant. During the last four decades, a number of duel games have been analyzed in the literature.

Dresher [4] and Karlin [9] introduced some results of simplest duel games in which each player has only one bullet. Restorepo [11] considered a silent-type game in which Player 1 possesses m bullets and Player 2 possesses n bullets. Fox and Kimeldorf [6] analyzed a noisy-type duel game, where each player has a different number of bullets in a fashion similar to the work by Restorepo [11]. Epstein [5] summarized features of strategies of the above duel games. Teraoka [17, 20] modeled the duel games where both players do not know whether there exist bullets in their gun or not. Teraoka [19] also dealt with a duel game where each duelist has an incomplete knowledge on the kind of opponent's bullet fitted to their guns. Kurisu [10] considered a duel game with equal accuracy functions. In his game, Player 1 has one noisy bullet and one silent bullet, and has to fire the noisy one first. On the other hand, Player 2 has only one silent bullet. Hendricks *et al.* [8] presented a general analysis of the war of attrition which is a type of game of timing in continuous time with complete information. More recently, the different models of the war of attrition were considered by Hamers [7] and Baston and Garnaev [2].

As an application of the game of timing, Sakaguchi [12] considered a cardinal game which is called marksmanship contest game. Since the seminal work by Sakaguchi [12], this type of game has been analyzed by some authors. Teraoka [21] modeled both silent-type and noisy-type marksmanship contest games with random termination for two players. The strategy which was derived as a solution of the game is the equilibrium strategy for both players, who have no motivation to change only their own strategies. However, it should be noted that both players have to choose their strategies, since the strategy derived by Teraoka [21] is an innumerable Nash equilibrium strategy. On the other hand, Baston and Garnaev [1] considered the similar but somewhat different silent-type game from [21] by taking a different expected total profit function in the case where both players fire at simultaneous time into consideration. Baston and Garnaev [1] showed that their game has a unique Nash equilibrium strategy. Furthermore, Teraoka [22] considered silent/noisy-type marksmanship contest games and derive their associated solutions.

In this paper, we formulate a unified marksmanship contest game which includes the previous games considered by Teraoka [21, 22] and Baston and Garnaev [1], referred to as silent-type Teraoka game, noisy-type Teraoka game, silent-type Baston and Garnaev game, as special cases. We show that our formulation splits into five different marksmanship contest games. The first four games are called silent-type Teraoka-Baston/Garnaev game, noisy-type Teraoka-Baston/Garnaev game, bonus game, and silent/noisy-type Teraoka-Baston/Garnaev game, and can be characterized as two-person nonzero-sum games of timing. The last game is the Stackelberg game [15] with two players; leader and follower. In this game, the follower has to wait for the leader's action, and reacts by choosing the best strategy consistent with the available information. It is well known that the Stackelberg game can be applied in several applications [3, 13, 14, 23]. In this paper we also characterize the Stackelberg strategy for the marksmanship contest game.

The remaining part of this paper is organized as follows. In Section 2, we give the mathematical preliminary for a fundamental marksmanship contest game with random termination and define the Nash equilibrium strategy. In Section 3, we summarize the existing results for three different marksmanship contest games (silent-type Teraoka game, noisy-type Teraoka game, silent-type Baston and Garnaev game) in [1, 21]. Section 4 is devoted to formulate a unified marksmanship contest game with random termination which splits into five different games (silent-type Teraoka-Baston/Garnaev game, noisy-type Teraoka-Baston/Garnaev game, bonus game, silent/noisy-type Teraoka-Baston/Garnaev game, Stackelberg game). We derive the solutions of respective marksmanship contest games analytically. In Section 5, we give simple numerical examples on the game solutions. Finally the paper is concluded with some remarks in Section 6.

2. Model Description

In the marksmanship contest game, suppose that there are two rival players in the marksmanship contest, and that each player has a gun with one bullet. We call them Player 1 and Player 2. The distance between each player and his or her own target is one unit length. Two players move to the target with unit speed, and fire at an arbitrary time (or an arbitrary point) $t \in [0, 1]$. The player who first hits the target is recognized as the winner, so the winner can receive unit reward from an umpire of the contest. Once the winner is decided, the contest ends. Here, we define the profit of each game when both players hit their own targets at simultaneous time individually. The accuracy function $A_i(t)$ $(i = 1, 2)$ represents the probability that Player i hits when he or she fires at time t. Let $A_i(t)$ be the function satisfying $A_i(0) = 0$ and $A_i(1) = 1$. Furthermore, suppose that this contest is to be terminated at a random time $T \in [0, 1]$ following the continuous probability distribution function $H(t)$ which satisfies $H(0) = 0$ and $H(1) = 1$. The functions $A_i(t)$ and $H(t)$ are continuous, strictly increasing and differentiable with respect to t. Of course, each player wishes to delay to fire in order to get the higher probability to win. However, they have to decide the shooting time taking the existence of rival player and random termination into consideration.

We define the following equation:

$$K_i(t) = \{1 - H(t)\}A_i(t), \quad i = 1, 2, \tag{2.1}$$

where $K_i(t)$ represents the probability that the game has not finished yet and Player i hits the target first. We assume that $K_i(t)$ is a unimodal function of t and has a unique maximum point m_i such as

$$m_i = \arg\max_{0 \le t \le 1} \{t | K_i(t)\}, \quad i = 1, 2. \tag{2.2}$$

There are two types of marksmanship contest game. If a player knows the time of opponent's action as soon as it takes place, then it is said that the opponent's player has a noisy bullet. In this paper, we call this game noisy-type game. On the other hand, if each player cannot know whether his or her opponent has acted or not, we say that both players have silent bullets. This game is called silent-type game.

For silent-type game, let (x, y) denote the pure strategies expressing the timings of actions for Player 1 and Player 2, respectively. Also, let $M_i(x, y)$ be the expected total profit for Player i $(= 1, 2)$ when Player 1 and Player 2 act at time x and y, respectively. Here, let $(X, Y) \in [0, 1] \times [0, 1]$ be the set of pure strategy for each Player i $(= 1, 2)$. Define the mixed strategy for each player:

$$F_1 = F_1(x) = \Pr\{X \le x\} \in [0, 1], \tag{2.3}$$
$$F_2 = F_2(y) = \Pr\{Y \le y\} \in [0, 1]. \tag{2.4}$$

This implies that Player i triggers the action at a random timing with the probability distribution function F_i. If Player i takes the mixed strategy F_i, then the expected total profit for Player i is given by

$$M_i(F_1, F_2) = \int_X \int_Y M_i(x, y) dF_1 dF_2, \tag{2.5}$$

where

$$M_1(x, F_2) = \int_Y M_1(x, y) dF_2, \tag{2.6}$$
$$M_2(F_1, y) = \int_X M_2(x, y) dF_1. \tag{2.7}$$

the set of mixed strategies (F_1^*, F_2^*) is called the *mixed equilibrium strategies* or *Nash equilibrium strategies* if they satisfy

$$M_1(F_1^*, F_2^*) \ge M_1(F_1, F_2^*), \tag{2.8}$$
$$M_2(F_1^*, F_2^*) \ge M_2(F_1^*, F_2) \tag{2.9}$$

for any mixed strategies F_1, F_2 on $[0, 1]$. The expected total profit $M_i(F_1^*, F_2^*)$ is called the *value function*.

On the other hand, for noisy-type game, let $(x, r_1(y))$ denotes the pure strategy for Player 1, such that Player 1 selects time x and then acts at time $r_1(y)$ if Player 2 has acted at time y before time x, or acts at time x if Player 2 has not yet acted before time x. Let $(y, r_2(x))$ be the pure strategy for Player 2, such that Player 2 selects time y and then acts at time $r_2(x)$ if Player 1 has acted at time x before time y, or acts at time y if Player 1 has not yet acted before time y. Let (F_i, r_i) be a mixed strategy such that Player i shoots at a random time with F_i if Player $3-i$ $(i = 1, 2)$ has not acted yet and shoots at a fixed time r_i if Player $3-i$ has already acted and failed under the condition that Player i has not acted yet. Nash equilibrium strategies and value functions can be defined in a similar way to silent-type game. Therefore, the challenge here is to find the Nash equilibrium strategies and their associated value functions.

3. Related Work

3.1. Teraoka's strategy for silent-type game

Teraoka [21] models a two-person game of timing with random termination which is called two-person nonzero-sum marksmanship contest game. For better understanding of our results, we summarize the results in [21]. Note that Teraoka [21] focuses on the both types of marksmanship contest game. In the silent-type game, Teraoka [21] formulates the expected total profit as follows.

$$M_1(x,y) = \begin{cases} K_1(x), & x \le y, \\ \{1 - A_2(y)\}K_1(x), & x > y, \end{cases} \tag{3.1}$$

$$M_2(x,y) = \begin{cases} K_2(y), & y \le x, \\ \{1 - A_1(x)\}K_2(y), & y > x. \end{cases} \tag{3.2}$$

In this paper, the game which has the expected total profit functions represented by Equations (3.1) and (3.2) is called Silent-type Teraoka game.

For any a satisfying $F_i(a) = 0$, let

$$f_i(t) = \frac{K_{3-i}(a)K'_{3-i}(t)}{\{K_{3-i}(t)\}^2 A_i(t)}, \quad a \le t \le m_{3-i}, \tag{3.3}$$

where $K'_i(t) = dK_i(t)/dt$. Define

$$m = \min(m_1, m_2), \tag{3.4}$$

and a_i $(i = 1, 2)$ is the unique root a of the equation:

$$\int_a^m f_i(t)dt = 1. \tag{3.5}$$

Teraoka [21] derives the following mixed strategy in the silent-type Teraoka game:

$$F_1^*(x) = \begin{cases} 0, & 0 \le x < a, \\ \int_a^x f_1(t)dt + \alpha_1^{[a,m]} I_m(x), & a \le x \le m, \\ 1, & m < x \le 1, \end{cases} \tag{3.6}$$

$$F_2^*(y) = \begin{cases} 0, & 0 \le y < a, \\ \int_a^y f_2(t)dt + \alpha_2^{[a,m]} I_m(y), & a \le y \le m, \\ 1, & m < y \le 1, \end{cases} \tag{3.7}$$

where

$$a = \max(a_1, a_2), \tag{3.8}$$

$$I_u(z) = \begin{cases} 1, & z = u, \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

$$\alpha_i^{[a,m]} = 1 - \int_a^m f_i(t)dt, \tag{3.10}$$

and the parameters $\alpha_i^{[a,m]}$ $(i = 1, 2)$ represent the mass part of Player i's mixed strategy. In [21], the value functions satisfying Equations (2.8) and (2.9) are given by

$$M_i(F_1^*, F_2^*) = K_i(a), \quad i = 1, 2. \tag{3.11}$$

So Teraoka [21] proves that the Nash equilibrium strategy which is shown by Equations (3.6) and (3.7), with the value functions in Equation (3.11) are the solution of the silent-type Teraoka game.

3.2. Teraoka's strategy for noisy-type game

In the noisy-type game, Teraoka [21] formulates the expected total profit as follows.

$$M_1((x, r_1(y)), (y, r_2(x))) = \begin{cases} K_1(x), & x \le y, \\ \{1 - A_2(y)\}K_1(r_1(y)), & x > y, \end{cases} \tag{3.12}$$

$$M_2((x, r_1(y)), (y, r_2(x))) = \begin{cases} K_2(y), & y \le x, \\ \{1 - A_1(x)\}K_2(r_2(x)), & y > x, \end{cases} \tag{3.13}$$

where $M_i((x, r_1(y)), (y, r_2(x)))$ is the expected total profit for Player i and $r_i(t)$ is the best reaction timing for Player i when Player $3 - i$ shoots at time t first. The game which has the expected total profit functions represented by Equations (3.12) and (3.13) is called Noisy-type Teraoka game. Note that each player can shoot at the best timing $r_i(t)$ in the situation where the opponent's player has already shot and failed.

Define

$$b = \max(b_1, b_2), \tag{3.14}$$

where b_i $(i = 1, 2)$ is the unique root $b \in [0, m_i]$ of the equation:

$$K_i(b) = \{1 - A_{3-i}(b)\}K_i(m_i), \quad i = 1, 2. \tag{3.15}$$

For any $t \in (b_{3-i}, m_{3-i}]$, we also define a function $\theta_i(t)$ as follows:

$$\theta_i(t) = \frac{K'_{3-i}(t)}{K_{3-i}(t) - \{1 - A_i(t)\}K_{3-i}(m_{3-i})}, \quad i = 1, 2. \tag{3.16}$$

Teraoka [21] derives the following function in $b < m$ for the noisy-type Teraoka game:

$$F_1^*(x) = \begin{cases} 0, & 0 \le x < b_0, \\ 1 - \exp\{-\int_{b_0}^x \theta_1(t)dt\} + \beta_1^{[b_0,c]} I_c(x), & b_0 \le x \le c, \\ 1, & c < x \le 1. \end{cases} \tag{3.17}$$

$$F_2^*(y) = \begin{cases} 0, & 0 \le y < b_0, \\ 1 - \exp\{-\int_{b_0}^y \theta_2(t)dt\} + \beta_2^{[b_0,c]} I_c(y), & b_0 \le y \le c, \\ 1, & c < y \le 1, \end{cases} \tag{3.18}$$

where

$$\beta_i^{[b_0,c]} = \exp\left\{-\int_{b_0}^{c} \theta_i(t)dt\right\}, \quad i=1,2, \tag{3.19}$$

and $[b_0, c]$ is any subset of $(b, m]$. With these functions, the Nash equilibrium strategy is given by a pair of $((F_1^*, m_1), (F_2^*, m_2))$, and the value functions are given by

$$M_i((F_1^*, m_1), (F_2^*, m_2)) = K_i(b_0), \quad i=1,2. \tag{3.20}$$

On the other hand, Teraoka [21] gives the Nash equilibrium strategy $((F_1^*, m_1), (F_2^*, m_2))$ in $b \geq m$ for the noisy-type Teraoka game, where

$$F_1^*(x) = \begin{cases} 0, & 0 \leq x < m_1, \\ 1, & m_1 \leq x \leq 1, \end{cases} \tag{3.21}$$

$$F_2^*(y) = \begin{cases} 0, & 0 \leq y < m_2, \\ 1, & m_2 \leq y \leq 1. \end{cases} \tag{3.22}$$

If $m = m_2$, then the value functions are given by

$$M_1((F_1^*, m_1), (F_2^*, m_2)) = \{1 - A_2(m_2)\}K_1(m_1), \tag{3.23}$$

$$M_2((F_1^*, m_1), (F_2^*, m_2)) = K_2(m_2). \tag{3.24}$$

3.3. Baston and Garnaev's strategy for silent-type game

Baston and Garnaev [1] consider a somewhat different game of the silent-type Teraoka game. In the silent-type game, they introduce a slightly different expected total profit as follows.

$$M_1(x,y) = \begin{cases} K_1(x), & x < y, \\ P_1(x), & x = y, \\ \{1 - A_2(y)\}K_1(x), & x > y, \end{cases} \tag{3.25}$$

$$M_2(x,y) = \begin{cases} K_2(y), & y < x, \\ P_2(y), & y = x, \\ \{1 - A_1(x)\}K_2(y), & y > x, \end{cases} \tag{3.26}$$

where $P_i(t)$ $(i = 1, 2)$ is the function satisfying the condition: $0 \leq P_i(t) < K_i(t)$. In this paper, the game which has the expected total profit functions represented by Equations (3.25) and (3.26) is called Silent-type Baston/Garnaev game. Compared with the silent-type Teraoka game, each player in the silent-type Baston/Garnaev game gains the cheaper profit in the situation where both players hit their own target at the same time.

Define $a_i^* \in [0, m_i]$ which is the unique root of the equation in the case of $m = m_i$:

$$K_{3-i}(m_{3-i}) \left[\frac{K_{3-i}(a_i^*)}{K_{3-i}(m_i)} - A_i(m_i)\left(1 - \int_{a_i^*}^{m_i} f_i(t)dt\right)\right] = K_{3-i}(a_i^*). \tag{3.27}$$

Baston and Garnaev [1] derive the following mixed strategy:

$$F_1^*(x) = \begin{cases} 0, & 0 \leq x < a, \\ \int_a^x f_1(t)dt, & a \leq x < m, \\ \int_a^m f_1(t)dt + \alpha_1^{[a,m]} I_{m_1}(x), & m \leq x \leq m_1, \\ 1, & m_1 < x \leq 1, \end{cases} \tag{3.28}$$

$$F_2^*(y) = \begin{cases} 0, & 0 \leq y < a, \\ \int_a^y f_2(t)dt, & a \leq y < m, \\ \int_a^m f_2(t)dt + \alpha_2^{[a,m]} I_{m_2}(y), & m \leq y \leq m_2, \\ 1, & m_2 < y \leq 1, \end{cases} \tag{3.29}$$

where

$$a = \begin{cases} \max(a_1, a_2^*), & m = m_2, \\ \max(a_2, a_1^*), & m = m_1. \end{cases} \tag{3.30}$$

In the above solution, the value functions satisfying Equations (2.8) and (2.9) are given by

$$M_i(F_1^*, F_2^*) = K_i(a), \quad i = 1, 2. \tag{3.31}$$

Baston and Garnaev [1] show that the silent-type Baston/Garnaev game has the unique Nash equilibrium strategy.

Remark 3.1. The Nash equilibrium strategy for the silent-type Teraoka game is a mixed strategy in which only one player has the mass part. On the other hand, the Nash equilibrium strategy for the silent-type Baston/Garnaev game is a mixed strategy in which both players might have the mass parts at different time m_i.

4. Unified Marksmanship Contest Game

4.1. Silent-type Teraoka-Baston/Garnaev game

Corresponding to the previous study [1, 21], we define $P_i(t)$ satisfying $0 \leq P_i(t) \leq K_i(t)$ for the game with Equations (3.25) and (3.26), and derive the Nash equilibrium strategy for a unified game which contains both Teraoka game and Baston/Garnaev game. This game is equivalent to the silent-type Baston/Garnaev game in the case of $0 \leq P_i(t) < K_i(t)$ and the silent-type Teraoka game in the case of $P_i(t) = K_i(t)$. The game which contains both games is called Silent-type Teraoka-Baston/Garnaev game in this paper. In Subsection 4.3 later, we will also consider the game in the case of $P_i(t) > K_i(t)$.

In Equations (2.6) and (2.7), the first-order conditions of optimality are given by

$$\frac{\partial}{\partial x} M_1(x, F_2) = 0, \quad 0 < x < m_1, \tag{4.1}$$

$$\frac{\partial}{\partial y} M_2(F_1, y) = 0, \quad 0 < y < m_2. \tag{4.2}$$

We suppose that there exists the first derivative of the mixed strategies F_i satisfying Equations (4.1) and (4.2), *i.e.*,

$$f_i(t) = \frac{dF_i(t)}{dt}, \quad 0 < t < m_i, \quad i = 1, 2. \tag{4.3}$$

Lemma 4.1. Let

$$\lambda_i(a) = K_{3-i}(m_{3-i}) \left[\frac{K_{3-i}(a)}{K_{3-i}(m_i)} - A_i(m_i) \left(1 - \int_a^{m_i} f_i(t) dt \right) \right] - K_{3-i}(a). \tag{4.4}$$

$\lambda_i(a)$ is a strictly decreasing function in the range of $0 \leq a \leq m_i$. Furthermore, there exists a unique root $a_i^* \in [0, m_i]$ of the equation: $\lambda_i(a) = 0$ under the condition of $\{1 - A_i(m_i)\} K_{3-i}(m_{3-i}) \leq K_{3-i}(m_i)$.

Proof. For $i = 1$ and $0 < a < m_1$, we have

$$\lambda_1'(a) \;\; = \;\; K_2'(a) \left[\frac{K_2(m_2)}{K_2(m_1)} - 1 - K_2(m_2) A_1(m_1) \right.$$

$$
\begin{aligned}
& \times \left(\int_a^{m_1} \left(\frac{1}{K_2(y)} \right)' \frac{1}{A_1(y)} dy + \frac{1}{K_2(a)} \frac{1}{A_1(a)} \right) \Bigg] \\
= & -K_2'(a) \left[1 + K_2(m_2) A_1(m_1) \int_a^{m_1} \frac{A_1'(y)}{K_2(y)\{A_1(y)\}^2} dy \right] \\
< & \ 0, \qquad (4.5)
\end{aligned}
$$

due to the integration by parts. Since

$$
\lambda_1(a_1) = K_2(a_1) \left[\frac{K_2(m_2)}{K_2(m_1)} - 1 \right] > 0, \qquad (4.6)
$$

$$
\lambda_1(m_1) = \{1 - A_1(m_1)\} K_2(m_2) - K_2(m_1) \leq 0, \qquad (4.7)
$$

the result follows and the proof is completed. □

Lemma 4.2. For the silent-type Teraoka-Baston/Garnaev game, the function which satisfies the first-order conditions of optimality is given by

$$
f_i^*(t) = \begin{cases} 0, & 0 \leq t < a, \\ \frac{K_{3-i}(a) K_{3-i}'(t)}{\{K_{3-i}(t)\}^2 A_i(t)}, & a \leq t < m, \\ 0, & m \leq t \leq 1. \end{cases} \qquad (4.8)
$$

Proof. When Player 1 shoots at time $x \in [a, m]$, we obtain

$$
\begin{aligned}
M_1(x, F_2^*) &= \int_a^m M_1(x, y) dF_2^* \\
&= K_1(x) \left[1 - \int_a^x A_2(y) dF_2^* \right]. \qquad (4.9)
\end{aligned}
$$

By differentiating Equation (4.9) with respect to x and setting it equal to zero, we have

$$
\frac{\left(1 - \int_a^x A_2(y) f_2^*(y) dy\right)'}{1 - \int_a^x A_2(y) f_2^*(y) dy} = -\frac{K_1'(x)}{K_1(x)}. \qquad (4.10)
$$

Integrating both sides of Equation (4.10) yields

$$
f_2^*(t) = \frac{d \cdot K_1'(t)}{\{K_1(t)\}^2 A_2(t)}, \qquad (4.11)
$$

where d is the constant of integration. By substituting $x = a$ into Equations (4.10) and (4.11), we have $d = K_1(a)$. The proof for Player 2 is made in the similar way. The proof is completed. □

Theorem 4.1. Let $m = m_{3-i}$. For the silent-type Teraoka-Baston/Garnaev game, under the condition that $\{1 - A_{3-i}(m_{3-i})\} K_i(m_i) \geq K_i(m_{3-i})$ holds, the Nash equilibrium strategy for each player is given by

$$
F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \qquad (4.12)
$$

$$
F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1. \end{cases} \qquad (4.13)
$$

The value functions for respective players are given by

$$M_i(F_1^*, F_2^*) = \{1 - A_{3-i}(m_{3-i})\}K_i(m_i), \tag{4.14}$$
$$M_{3-i}(F_1^*, F_2^*) = K_{3-i}(m_{3-i}). \tag{4.15}$$

Proof. The function $K_i(m_{3-i})$ represents the expected total profit for Player i in the situation where he or she hits the target at time m_{3-i} before the opponent's action. Also, the function $\{1 - A_{3-i}(m_{3-i})\}K_i(m_i)$ denotes the expected total profit for Player i when he or she shoots at time m_i after the opponent's fire at time m_{3-i}. Therefore, if the condition that $\{1 - A_{3-i}(m_{3-i})\}K_i(m_i) \geq K_i(m_{3-i})$ holds, then Player $3 - i$ wishes to fire at time m_{3-i}, and Player i selects to shoot at time m_i after the opponent's action. It is easy to check that this strategy satisfies the Nash inequalities in Equations (2.8) and (2.9). □

Theorem 4.2. Let $m = m_{3-i}$. For the silent-type Teraoka-Baston/Garnaev game, under the condition that $\{1-A_{3-i}(m_{3-i})\}K_i(m_i) < K_i(m_{3-i})$ holds, the Nash equilibrium strategy for each player is given by

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < a, \\ \int_a^t f_i^*(t)dt, & a \leq t < m, \\ \int_a^m f_i^*(t)dt + \gamma_i^{[a,m]} I_{m_i}(z), & m \leq t \leq m_i, \\ 1, & m_i < t \leq 1, \end{cases} \tag{4.16}$$

where $a = \max(a_{3-i}^*, a_i)$, and $\gamma_i^{[a,m]}$ $(i = 1, 2)$ which represents the mass part, is given by

$$\gamma_i^{[a,m]} = 1 - \int_a^m f_i^*(t)dt. \tag{4.17}$$

The value functions of both players are given by

$$M_i(F_1^*, F_2^*) = K_i(a), \quad i = 1, 2. \tag{4.18}$$

Proof. For $i = 1$ and Player 1, we consider four cases; (i) $0 \leq x < a$, (ii) $a \leq x < m$, (iii) $x = m$, (iv) $m < x \leq 1$. In Case (i), from Equations (2.6) and (3.25), we have

$$M_1(x, F_2^*) = K_1(x) < K_1(a). \tag{4.19}$$

In Case (ii),

$$M_1(x, F_2^*) = \int_a^x \{1 - A_2(y)\}K_1(x)dF_2^* + \int_x^{m-0} K_1(x)dF_2^* + K_1(x)\gamma_2^{[a,m]}. \tag{4.20}$$

The function $M_1(x, F_2^*)$ is constant for the range of $a \leq x < m$, since F_2^* satisfies the first-order conditions of optimality. Substituting $x = a$ into Equation (4.20) leads to

$$M_1(a, F_2^*) = K_1(a) \left(\int_a^{m-0} dF_2^* + \gamma_2^{[a,m]} \right) = K_1(a). \tag{4.21}$$

In Case (iii), from Equation (4.16), we obtain $M_1(x, F_2^*)$ as follows.

$$M_1(m, F_2^*) = \int_a^{m-0} \{1 - A_2(y)\}K_1(m)dF_2^* + P_1(m)\gamma_2^{[a,m]}$$

$$
\begin{aligned}
&= K_1(a) - \{K_1(m) - P_1(m)\}\,\gamma_2^{[a,m]} \\
&\le K_1(a).
\end{aligned}
\tag{4.22}
$$

In Case (iv), we have

$$
\begin{aligned}
M_1(x, F_2^*) &= \int_a^{m-0} \{1 - A_2(y)\} K_1(x) dF_2^* + \{1 - A_2(m)\} K_1(x) \gamma_2^{[a,m]} \\
&= K_1(x) \left\{ \frac{K_1(a)}{K_1(m)} - A_2(m) \gamma_2^{[a,m]} \right\}.
\end{aligned}
\tag{4.23}
$$

From the property of $K_1(x)$, Equation (4.23) has the maximum value at $x = m_1$, and $M_1(m_1, F_2^*)$ is given by

$$
M_1(m_1, F_2^*) = K_1(m_1) \left\{ \frac{K_1(a)}{K_1(m)} - A_2(m) \gamma_2^{[a,m]} \right\}. \tag{4.24}
$$

We show that Equation (4.24) is not greater than $K_1(a)$. In the case of $a = a_2^*$, from Lemma 4.1, we obtain

$$
M_1(m_1, F_2^*) = K_1(m_1) \left\{ \frac{K_1(a_2^*)}{K_1(m)} - A_2(m) \gamma_2^{[a,m]} \right\} = K_1(a_2^*). \tag{4.25}
$$

On the other hand, in the case of $a = a_1$, since $\lambda(a_1) < 0$ from Lemma 4.1, we have

$$
M_1(m_1, F_2^*) = K_1(m_1) \left\{ \frac{K_1(a_1)}{K_1(m)} - A_2(m) \gamma_2^{[a,m]} \right\} < K_1(a_1). \tag{4.26}
$$

From the above cases (i)-(iv), it is shown for all x that

$$
M_1(x, F_2^*) \le K_1(a). \tag{4.27}
$$

In the case of $a = a_2^*$, the game value is given by

$$
M_1(F_1^*, F_2^*) = \int_{a_2^*}^{m-0} K_1(a_2^*) dF_1^* + K_1(a_2^*) \gamma_1^{[a,m]} = K_1(a_2^*). \tag{4.28}
$$

Finally, in the case of $a = a_1$, $\gamma_1^{[a,m]}$ equals to zero from Equations (3.5) and (4.17), so we find

$$
M_1(F_1^*, F_2^*) = \int_{a_1}^{m-0} K_1(a_1) dF_1^* = K_1(a_1). \tag{4.29}
$$

Since the case for Player 2 is similar, the proof is completed. □

It can be easily checked that the strategies which are shown in Theorem 4.1 and Theorem 4.2 are the Nash equilibrium strategy for silent-type Baston/Garnaev game. Since the silent-type Baston/Garnaev game has a unique Nash equilibrium strategy, it can be easily checked that these solutions are also unique Nash equilibrium strategy for silent-type Teraoka-Baston/Garnaev game.

4.2. Noisy-type Teraoka-Baston/Garnaev game

In this subsection, we consider the noisy-type game. Define the expected total profit of this game as follows.

$$M_1((x,r_1(y)),(y,r_2(x))) = \begin{cases} K_1(x), & x<y, \\ P_1(x), & x=y, \\ \{1-A_2(y)\}K_1(r_1(y)), & x>y, \end{cases} \tag{4.30}$$

$$M_2((x,r_1(y)),(y,r_2(x))) = \begin{cases} K_2(y), & y<x, \\ P_2(y), & y=x, \\ \{1-A_1(x)\}K_2(r_2(x)), & y>x. \end{cases} \tag{4.31}$$

Similar to Subsection 4.1, we define $P_i(t)$ satisfying $0 \le P_i(t) \le K_i(t)$ in Equations (4.30) and (4.31). Note that Baston and Garnaev [1] did not consider the noisy-type game. Hereafter, we call the game which has above conditions Noisy-type Teraoka-Baston/Garnaev game in this paper. In a similar fashion to the silent-type game, we will consider the game in the case of $P_i(t) > K_i(t)$ in Subsequent 4.3.

For the noisy-type game, the first-order conditions of optimality are given by

$$\frac{\partial}{\partial x} M_1((x,r_1(y)),(F_2,r_2(x))) = 0, \quad 0<x<m_1, \tag{4.32}$$

$$\frac{\partial}{\partial y} M_2((F_1,r_1(y)),(y,r_2(x))) = 0, \quad 0<y<m_2. \tag{4.33}$$

Lemma 4.3. For the noisy-type Teraoka-Baston/Garnaev game, the strategy $F_i^*(t)$ $(i = 1,2)$ which satisfies the first-order conditions of optimality is given by

$$F_i^*(t) = 1 - \exp\left\{ -\int_b^t \theta_i(t)dt \right\}, \quad b<t<c, \tag{4.34}$$

where $b = \max(b_1, b_2)$ and c is an arbitrary real number satisfying $b < c \le m$.

Proof. When Player 1 shoots at $x \in [b,c]$, we obtain

$$M_1((x,r_1(y)),(F_2^*,r_2(x))) = \int_b^x \{1-A_2(y)\}K_1(m_1)dF_2^* + \int_x^c K_1(x)dF_2^*. \tag{4.35}$$

From Equation (4.32), we have

$$\frac{F_2^{*\prime}(x)}{1-F_2^*(x)} = \frac{K_1'(x)}{K_1(x) - \{1-A_2(x)\}K_1(m_1)} = \theta_2(x). \tag{4.36}$$

Integrating both sides of Equation (4.36) with respect to x yields

$$F_2^*(t) = 1 - \eta \exp\left\{ -\int_b^t \theta_2(t)dt \right\}, \tag{4.37}$$

where η is the constant of integration. By substituting $t = b$ into Equation (4.37), we get $\eta = 1$. The proof for Player 2 is made in the similar way. The proof is completed. □

Lemma 4.4. For an arbitrary $c \in (b, m]$, the following equation holds:

$$\int_a^c \theta_{3-i}(t)dt \uparrow \infty, \quad \text{as } a \downarrow b_i. \tag{4.38}$$

Proof. See Sweat [16] and Teraoka [18]. □

Theorem 4.3. Let $m = m_{3-i}$. For the noisy-type Teraoka-Baston/Garnaev game, under the condition that $b \geq m$ holds, the Nash equilibrium strategies for respective players are given by a pair of $((F_1^*, m_1), (F_2^*, m_2))$, where

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \tag{4.39}$$

$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1. \end{cases} \tag{4.40}$$

The value functions for respective players are given by

$$M_i((F_1^*, m_1), (F_2^*, m_2)) = \{1 - A_{3-i}(m_{3-i})\}K_i(m_i), \tag{4.41}$$

$$M_{3-i}((F_1^*, m_1), (F_2^*, m_2)) = K_{3-i}(m_{3-i}). \tag{4.42}$$

The proof is omitted for brevity from the similar argument in Theorem 4.1.

Theorem 4.4. Let $b = b_{3-i}(\neq b_i)$. For the noisy-type Teraoka-Baston/Garnaev game, under the condition that $b < m$ holds, the Nash equilibrium strategies for respective players are given by $((F_1^*, m_1), (F_2^*, m_2))$, where

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < b, \\ 1, & b \leq t \leq 1, \end{cases} \tag{4.43}$$

$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < b, \\ 1 - \exp\{-\int_b^t \theta_{3-i}(t)dt\} + \beta_{3-i}^{[b,c]} I_c(t), & b \leq t \leq c, \\ 1, & c < t \leq 1, \end{cases} \tag{4.44}$$

and $c \leq m$. The value functions of both players are given by

$$M_i((F_1^*, m_1), (F_2^*, m_2)) = K_i(b), \quad i = 1, 2. \tag{4.45}$$

Proof. In the case where Player 2 takes the Nash equilibrium strategy, we calculate the expected total profit for Player 1 as follows.

$$\begin{aligned} &M_1((x, r_1(y)), (F_2^*, m_2)) \\ &= \begin{cases} K_1(x), & 0 \leq x < b, \\ K_1(b), & b \leq x < c, \\ K_1(b) - \{K_1(c) - P_1(c)\}\, \beta_2^{[b,c]}, & x = c, \\ K_1(b) - [K_1(c) - \{1 - A_2(c)\}\, K_1(m_1)]\, \beta_2^{[b,c]}, & c < x \leq 1. \end{cases} \end{aligned} \tag{4.46}$$

Also, in the case where Player 1 takes the Nash equilibrium strategy, the expected total profit for Player 2 is given by

$$M_2((F_1^*, m_1), (y, r_2(x))) = \begin{cases} K_2(y), & 0 \leq y < b, \\ P_2(b), & y = b, \\ \{1 - A_1(b)\}K_2(m_2) = K_2(b), & b < y \leq c, \\ K_2(b), & c < y \leq 1. \end{cases} \tag{4.47}$$

Hence, it is shown for all x and y that

$$\begin{aligned} M_1((x, r_1(y)), (F_2^*, m_2)) &\leq K_1(b), & (4.48) \\ M_2((F_1^*, m_1), (y, r_2(x))) &\leq K_2(b). & (4.49) \end{aligned}$$

The value functions of both players are given by

$$M_i((F_1^*, m_1), (F_2^*, m_2)) = K_i(b), \quad i = 1, 2. \tag{4.50}$$

The proof is completed. □

Theorem 4.5. Let $b = b_{3-i} (\neq b_i)$ and $m = m_i$. For the noisy-type Teraoka-Baston/Garnaev game, under the condition that $b < m$ holds, the Nash equilibrium strategies for respective players are given by $((F_1^*, m_1), (F_2^*, m_2))$, where

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < b, \\ 1, & b \leq t \leq 1, \end{cases} \tag{4.51}$$

$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < b, \\ 1 - \exp\{-\int_b^t \theta_{3-i}(t)dt\}, & b \leq t < m, \\ 1 - \exp\{-\int_b^m \theta_{3-i}(t)dt\} + \beta_{3-i}^{[b,m]} I_d(t), & m \leq t \leq d, \\ 1, & d < t \leq 1, \end{cases} \tag{4.52}$$

and d is an arbitrary real number satisfying $m < d \leq 1$. The value functions of both players are given by

$$M_i((F_1^*, m_1), (F_2^*, m_2)) = K_i(b), \quad i = 1, 2. \tag{4.53}$$

Proof. For $i = 1$, the expected total profit for Player 1 is given by

$$\begin{aligned} &M_1((x, r_1(y)), (F_2^*, m_2)) \\ &= \begin{cases} K_1(x), & 0 \leq x < b, \\ K_1(b), & b \leq x \leq m, \\ K_1(b) - \{K_1(m_1) - K_1(x)\} \beta_2^{[b,m]}, & m < x < d, \\ K_1(b) - \{K_1(m_1) - P_1(d)\} \beta_2^{[b,m]}, & x = d, \\ K_1(b) - [K_1(m_1) - \{1 - A_2(d)\} K_1(d)] \beta_2^{[b,m]}, & d < x \leq 1. \end{cases} \end{aligned} \tag{4.54}$$

On the other hand, the expected total profit for Player 2 is also given by

$$M_2((F_1^*, m_1), (y, r_2(x))) = \begin{cases} K_2(y), & 0 \leq y < b, \\ P_2(b), & y = b, \\ \{1 - A_1(b)\} K_2(m_2) = K_2(b), & b < y < m, \\ K_2(b), & m \leq y < d, \\ K_2(b), & d \leq y \leq 1. \end{cases} \tag{4.55}$$

Hence, it can be shown for all x and y that

$$\begin{aligned} M_1((x, r_1(y)), (F_2^*, m_2)) &\leq K_1(b), & (4.56) \\ M_2((F_1^*, m_1), (y, r_2(x))) &\leq K_2(b). & (4.57) \end{aligned}$$

The value functions of both players are given by

$$M_i((F_1^*, m_1), (F_2^*, m_2)) = K_i(b), \quad i = 1, 2. \tag{4.58}$$

The proof is completed. □

The condition $b \geq m$ in Theorem 4.3 is equivalent to the condition $\{1-A_{3-i}(m_{3-i})\}K_i(m_i) \geq K_i(m_{3-i})$ in Theorem 4.1. shown in Theorem 4.3. Therefore, under the same condition, the strategies that Player 1 and Player 2 always shoot at time m_1 and m_2 respectively are the Nash equilibrium strategies in both silent-type game and noisy-type game. In the silent-type Teraoka-Baston/Garnaev game, we derive the Nash equilibrium strategy as a mixed strategy. On the other hand, in the noisy-type Teraoka-Baston/Garnaev game, there exists the Nash equilibrium strategy in which one player takes a pure strategy and the other player takes a mixed strategy as shown in Theorem 4.4 and Theorem 4.5. Furthermore, we can see that the Nash equilibrium strategy in the noisy-type Teraoka-Baston/Garnaev game is not a unique solution unlike in the case of the silent-type Teraoka-Baston/Garnaev game. In the above discussion, we did not give the solutions in the case $b_1 = b_2$ of noisy-type Teraoka-Baston/Garnaev game. However, this situation occurs only in a very limited case where two players have the absolutely same accuracy function.

4.3. Bonus game

The results of Subsection 4.1 and Subsection 4.2 concern the solutions of the game with the condition of $0 \leq P_i(t) \leq K_i(t)$. Here, we derive the solutions of games with the expected total profits represented by Equations (3.25), (3.26), (4.30) and (4.31) under the condition of $P_i(t) > K_i(t)$. The former game is called Silent-type bonus game and the latter game is called Noisy-type bonus game. The strategy given in Theorem 4.6 and Theorem 4.7 holds regardless of the type of games. Therefore, we focus on only the silent-type strategy in this subsection. Suppose that $P_i(t)$ is a unimodal function which is maximum at time m_i, similar to the function $K_i(t)$ $(P_i(0) = P_i(1) = 0)$. Define the parameters ρ_i and σ_i as follows.

$$\rho_i = \inf\{t : K_i(m_i) = P_i(t)\}, \tag{4.59}$$
$$\sigma_i = \sup\{t : K_i(m_i) = P_i(t)\}. \tag{4.60}$$

Also define $\rho = \max(\rho_1, \rho_2)$ and $\sigma = \min(\sigma_1, \sigma_2)$.

Theorem 4.6. Let $m = m_{3-i}$. For the bonus game in the case of $b \geq m$, the Nash equilibrium strategies are given as follows.

(i) If $\{1 - A_{3-i}(m_{3-i})\}K_i(m_i) \geq P_i(m_{3-i})$ and $K_{3-i}(m_{3-i}) \geq P_{3-i}(m_i)$ hold, then the Nash equilibrium strategies for respective players are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \tag{4.61}$$
$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.62}$$

and the value functions are given by

$$M_i(F_1^*, F_2^*) = \{1 - A_{3-i}(m_{3-i})\}K_i(m_i), \tag{4.63}$$
$$M_{3-i}(F_1^*, F_2^*) = K_{3-i}(m_{3-i}). \tag{4.64}$$

(ii) If $\{1 - A_{3-i}(m_{3-i})\}K_i(m_i) \geq P_i(m_{3-i})$ and $K_{3-i}(m_{3-i}) < P_{3-i}(m_i)$ hold, then the Nash equilibrium strategies and their associated value functions are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \tag{4.65}$$
$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \tag{4.66}$$

$$M_i(F_1^*, F_2^*) = P_i(m_i), \tag{4.67}$$
$$M_{3-i}(F_1^*, F_2^*) = P_{3-i}(m_i). \tag{4.68}$$

(iii) If $\{1 - A_{3-i}(m_{3-i})\}K_i(m_i) < P_i(m_{3-i})$ and $K_{3-i}(m_{3-i}) \geq P_{3-i}(m_i)$ hold, then the Nash equilibrium strategies and their associated value functions are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.69}$$
$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.70}$$

$$M_i(F_1^*, F_2^*) = P_i(m_{3-i}), \tag{4.71}$$
$$M_{3-i}(F_1^*, F_2^*) = P_{3-i}(m_{3-i}). \tag{4.72}$$

(iv) If $\{1 - A_{3-i}(m_{3-i})\}K_i(m_i) < P_i(m_{3-i})$ and $K_{3-i}(m_{3-i}) < P_{3-i}(m_i)$ hold, then the Nash equilibrium strategies and their associated value functions are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \tag{4.73}$$
$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_i, \\ 1, & m_i \leq t \leq 1, \end{cases} \tag{4.74}$$

$$M_i(F_1^*, F_2^*) = P_i(m_i), \tag{4.75}$$
$$M_{3-i}(F_1^*, F_2^*) = P_{3-i}(m_i), \tag{4.76}$$

or

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.77}$$
$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.78}$$

$$M_i(F_1^*, F_2^*) = P_i(m_{3-i}), \tag{4.79}$$
$$M_{3-i}(F_1^*, F_2^*) = P_{3-i}(m_{3-i}). \tag{4.80}$$

Theorem 4.7. Let $m = m_{3-i}$. For the bonus game in the case of $b < m$, the Nash equilibrium strategies are given as follows.

(i) If $K_{3-i}(m_{3-i}) \geq P_{3-i}(m_i)$ holds, then the Nash equilibrium strategies and their associated value functions are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.81}$$
$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \leq t < m_{3-i}, \\ 1, & m_{3-i} \leq t \leq 1, \end{cases} \tag{4.82}$$

$$M_i(F_1^*, F_2^*) = P_i(m_{3-i}), \quad (4.83)$$
$$M_{3-i}(F_1^*, F_2^*) = P_{3-i}(m_{3-i}). \quad (4.84)$$

(ii) If $K_{3-i}(m_{3-i}) < P_{3-i}(m_i)$ holds, then the Nash equilibrium strategies and their associated value functions are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \le t < t^*, \\ 1, & t^* \le t \le 1, \end{cases} \quad (4.85)$$

$$F_{3-i}^*(t) = \begin{cases} 0, & 0 \le t < t^*, \\ 1, & t^* \le t \le 1, \end{cases} \quad (4.86)$$

$$M_i(F_1^*, F_2^*) = P_i(t^*), \quad (4.87)$$
$$M_{3-i}(F_1^*, F_2^*) = P_{3-i}(t^*), \quad (4.88)$$

where t^* is an arbitrary time satisfying $\rho \le t^* \le \sigma$.

Theorem 4.8. Let $m = m_1 = m_2$ and $a = a_1 = a_2$. For the silent-type bonus game, under the condition that $b < m$ holds, the Nash equilibrium strategies and their associated value functions are given by

$$F_i^*(t) = \begin{cases} 0, & 0 \le t < a, \\ \int_a^t f_i^*(t)dt, & a \le t < m, \\ 1, & m \le t \le 1, \end{cases} \quad (4.89)$$

$$M_i(F_1^*, F_2^*) = K_i(a), \quad i = 1, 2, \quad (4.90)$$

where

$$f_i^*(t) = \frac{K_{3-i}(a)K'_{3-i}(t)}{\{K_{3-i}(t)\}^2 A_i(t)}, \quad i = 1, 2. \quad (4.91)$$

It turns out that the solutions given in Theorem 4.6, Theorem 4.7 and Theorem 4.8 are actually the Nash equilibrium strategies. Therefore, the proofs of Theorem 4.6, Theorem 4.7 and Theorem 4.8 are omitted. In the bonus game, each player gets the more profit when both players shoot at the same time, so each player tends to take a pure strategy which is a deterministic action at the same time. In the case (iv) of Theorem 4.6, both strategies which are shown by Equations (4.73), (4.74) and Equations (4.77), (4.78) are the Nash equilibrium strategies. Also, all strategies that both players fire at the same time t^* satisfying $\rho \le t^* \le \sigma$ are the Nash equilibrium strategies as shown in the case (ii) of Theorem 4.7. It is clearly shown that there exists innumerable Nash equilibrium strategies. Furthermore, we can show the Nash equilibrium strategy which is a mixed strategy for the silent-type bonus game in Theorem 4.8. However, The solution of Theorem 4.8 holds in a very limited case where the parameters a_i $(i = 1, 2)$ and m_i $(i = 1, 2)$ are the same values.

4.4. Silent/Noisy-type Teraoka-Baston/Garnaev game

In this subsection, we consider the silent/noisy-type game. Suppose that Player 1 has a silent bullet and Player 2 has a noisy bullet. Therefore, Player 1 notices the opponent's action as soon as it takes place. On the other hand, Player 2 cannot know whether Player 1 has acted or not. Let $((x, r_1(y)), y)$ denote the pure strategies for Player 1 and Player 2. Define the expected total profit of this game as follows.

$$M_1((x, r_1(y)), y) = \begin{cases} K_1(x), & x < y, \\ P_1(x), & x = y, \\ \{1 - A_2(y)\}K_1(r_1(y)), & x > y, \end{cases} \tag{4.92}$$

$$M_2((x, r_1(y)), y) = \begin{cases} K_2(y), & y < x, \\ P_2(y), & y = x, \\ \{1 - A_1(x)\}K_2(y), & y > x, \end{cases} \tag{4.93}$$

where $M_i((x, r_1(y)), y)$ means the expected total profit for Player i, and $P_i(t)$ satisfies $0 \leq P_i(t) \leq K_i(t)$. Teraoka [22] considers the silent/noisy-type marksmanship contest game with random termination when $P_i(t) = K_i(t)$. We call this game with the above condition Silent/Noisy-type Teraoka-Baston/Garnaev game in this paper.

Theorem 4.9. For the silent/noisy-type Teraoka-Baston/Garnaev game, under the condition that $b < m$ holds, the Nash equilibrium strategies and the value functions for respective players are given by $((F_1^*, m_1), F_2^*)$ and $M_i((F_1^*, m_1), F_2^*)$, where

$$F_1^*(x) = \begin{cases} 0, & 0 \leq x < a, \\ \int_a^x f_1^*(t)dt, & a \leq x < m, \\ \int_a^m f_1^*(t)dt + \gamma_1^{[a,m]} I_{m_1}(z), & m \leq x \leq m_1, \\ 1, & m_1 < x \leq 1, \end{cases} \tag{4.94}$$

$$F_2^*(y) = \begin{cases} 0, & 0 \leq y < a, \\ 1 - \exp\{-\int_a^y \theta_2(t)dt\}, & a \leq y < m, \\ 1 - \exp\{-\int_a^m \theta_2(t)dt\} + \beta_2^{[a,m]} I_{m_2}(y), & m \leq y \leq m_2, \\ 1, & m_2 < y \leq 1, \end{cases} \tag{4.95}$$

$$M_i((F_1^*, m_1), F_2^*) = K_i(a), \quad i = 1, 2, \tag{4.96}$$

and

$$a = \begin{cases} \max(a_1^*, b_1), & m = m_1, \\ \max(a_1, b_1), & m = m_2. \end{cases} \tag{4.97}$$

It is evident to see that the solutions in Theorem 4.9 are actually the Nash equilibrium strategies. Therefore, the proof is omitted.

Theorem 4.10. Let $m = m_2$. For the silent/noisy-type Teraoka-Baston/Garnaev game, under the condition that $b_1 \geq m_2$ holds, the Nash equilibrium strategies for respective players are given by $((F_1^*, m_1), F_2^*)$, where

$$F_1^*(x) = \begin{cases} 0, & 0 \leq x < m_1, \\ 1, & m_1 \leq x \leq 1, \end{cases} \tag{4.98}$$

$$F_2^*(y) = \begin{cases} 0, & 0 \leq y < m_2, \\ 1, & m_2 \leq y \leq 1. \end{cases} \tag{4.99}$$

The value functions for respective players are given by

$$M_1((F_1^*, m_1), F_2^*) = \{1 - A_2(m_2)\}K_1(m_1), \quad (4.100)$$
$$M_2((F_1^*, m_1), F_2^*) = K_2(m_2). \quad (4.101)$$

We omit the proof, since we have already shown the similar results in Theorem 4.1.

4.5. Stackelberg strategy solution for marksmanship contest game with random termination

In this subsection, we consider a Stackelberg marksmanship contest game with random termination. In this game, we assume that there are leader and follower players. That is, there are superiority and inferiority between two players. Suppose that the leader can shoot earlier than the follower, and that the follower can act only at the same timing as the leader's shot or after. In this game, the expected total profit functions are given by Equations (3.25) and (3.26). We focus on the case of $P_i(t) = K_i(t)$.

Let Player 1 and Player 2 be the follower and the leader, respectively. For an arbitrary $y \in [0, 1]$, we define the function $f : [0, 1] \to [0, 1]$ satisfying the condition:

$$M_1(f(y), y) = \sup_{x \geq y} M_1(x, y). \quad (4.102)$$

If there exists an arbitrary $y^* \in [0, 1]$ which satisfies the condition:

$$M_2(f(y^*), y^*) = \sup_{y \in [0,1]} M_2(f(y), y), \quad (4.103)$$

then the Stackelberg strategies are given by (x^*, y^*), where $x^* = f(y^*)$. In the case where Player 1 is the leader, the Stackelberg strategies are given by a pair of $(\bar{x}^*, \bar{y}^*)$. It is immediate to see that

$$M_2(f(y^*), y^*) \geq M_2(F_1^*, F_2^*), \quad (4.104)$$

so, the leader can obtain the expected profit not less than the value function under the Nash equilibrium strategy. This means that the Stackelberg game is more profitable for the leader player.

Let J_j^i be the expected profit of Player j $(= 1, 2)$ in the case where Player i $(= 1, 2)$ is the leader, so we define $J_j^1 = M_j(\bar{x}^*, \bar{y}^*)$ and $J_j^2 = M_j(x^*, y^*)$.

Theorem 4.11. The Stackelberg strategies for Player 2 as the leader are given by

$$x^* = \begin{cases} m_1, & b_1 \geq m_2, \\ m_2, & b_1 < m_2, \end{cases} \quad (4.105)$$
$$y^* = m_2, \quad (4.106)$$

where b_1 and m_2 are already defined in Equations (3.15) and (2.2), respectively. The optimal payoffs for respective players are given by

$$M_1(x^*, y^*) = \begin{cases} \{1 - A_2(m_2)\}K_1(m_1), & b_1 \geq m_2, \\ K_1(m_2), & b_1 < m_2, \end{cases} \quad (4.107)$$
$$M_2(x^*, y^*) = K_2(m_2). \quad (4.108)$$

Theorem 4.12. In the Stackelberg marksmanship contest game with random termination, the following conditions always hold.

$$J_1^1 > J_1^2, \tag{4.109}$$
$$J_2^2 > J_2^1. \tag{4.110}$$

Therefore, the Stackelberg marksmanship contest game with random termination is competitive in the sense that both players wish to become the leader.

Remark 4.1. The Nash equilibrium strategies in Theorem 4.10 are equivalent to the solutions (x^*, y^*) in Theorem 4.11. That is, under the condition of $b_1 \geq m_2$, the actions of the leader and the follower in the Stackelberg game correspond to those of the noisy and silent players in the silent/noisy-type game. In the Nash equilibrium strategy, Player 1 does not have motivation to shot before time b_1. On the other hand, Player 2 always wishes to shot at time m_2. Consequently, the order of actions for two players is uniquely determined.

5. Numerical Examples

In this section, we give simple numerical examples. First, we derive the Nash equilibrium strategies and value functions for both silent-type and noisy-type Teraoka-Baston/Garnaev games. The model parameters for two players are given in Table 1. For example, we label the game with the condition: $A_1(t) = t, A_2(t) = t, H(t) = t$ as Case 1 as shown in Table 1. Here, the accuracy function $A_i(t)$ represents the probability that Player i hits his or her own target when Player i fires at time t. Therefore, we assume that the player who has the good accuracy function; $A_i(t) = -(t-1)^2 + 1$ can be regarded as an advanced contestant. If the player has the normal accuracy function; $A_i(t) = t$, then he or she is an intermediate contestant. Furthermore, the player with the poor accuracy function; $A_i(t) = t^2$ is regarded as a beginner. Also, it is seen that the game with the condition: $H(t) = -(t-1)^2 + 1$ is easy to terminate. On the other hand, the condition: $H(t) = t^2$ means that this game tends to continue further more.

Table 2 presents the support range and value function in the silent-type Teraoka-Baston/Garnaev game, where the underline denotes the supports of a and m. From Table 1 and Table 2, we can see that Player 2 has always better skills of shooting than Player 1. Therefore, the value function of Player 2 is greater than that of Player 1. Furthermore, we find that both players tend to get more profit if the game has the good condition relevant to the function $H(t)$ from Case 1, Case 3 and Case 7 of Table 2. Figure 1 shows the behavior of the Nash equilibrium strategy in the silent-type Teraoka-Baston/Garnaev game with parameters in Case 2. From this figure, we observe that only one player takes the Nash equilibrium strategy with probability mass part in the silent-type Teraoka-Baston/Garnaev game.

Table 3 presents the support range and value function in the noisy-type Teraoka-Baston/Garnaev game, where the underline denotes the supports of b and m. From Table 2 and Table 3, it is seen that the value functions of the noisy-type game are always greater than those of the silent-type game in the same case. These results arise from superiority condition of the noisy-type game for two players. As mentioned in Subsection 4.2, we did not give the solutions in the case $b_1 = b_2$ of the noisy-type game. Therefore, we show only the results when $b_1 \neq b_2$ in Table 3. Figure 2 shows the behavior of the Nash equilibrium strategy in the noisy-type Teraoka-Baston/Garnaev game with parameters in Case 2, where $c = m - 0.05$.

Table 1: Parameter setting

	$A_1(t)$	$A_2(t)$	$H(t)$
Case 1	t	t	t
Case 2	t^2	t	t
Case 3	t	t	t^2
Case 4	t^2	t	t^2
Case 5	t^2	t^2	t^2
Case 6	t	$-(t-1)^2+1$	t
Case 7	t	t	$-(t-1)^2+1$
Case 8	t	$-(t-1)^2+1$	$-(t-1)^2+1$
Case 9	$-(t-1)^2+1$	$-(t-1)^2+1$	$-(t-1)^2+1$
Case 10	t	$-(t-1)^2+1$	t^2
Case 11	t^2	$-(t-1)^2+1$	t
Case 12	t^2	t	$-(t-1)^2+1$

Table 2: Support range and value function in the silent-type Teraoka-Baston/Garnaev game

	a_1	a_2	m_1	m_2	$K_1(a)$	$K_2(a)$
Case 1	$\underline{0.2282}$	$\underline{0.2282}$	$\underline{0.5000}$	$\underline{0.5000}$	0.1761	0.1761
Case 2	0.3163	$\underline{0.3373}$	0.6667	$\underline{0.5000}$	0.0754	0.2235
Case 3	$\underline{0.2705}$	$\underline{0.2705}$	$\underline{0.5774}$	$\underline{0.5774}$	0.2507	0.2507
Case 4	0.3639	$\underline{0.3785}$	0.7071	$\underline{0.5774}$	0.1227	0.3243
Case 5	$\underline{0.4777}$	$\underline{0.4777}$	$\underline{0.7071}$	$\underline{0.7071}$	0.1761	0.1761
Case 6	$\underline{0.1989}$	0.1716	0.5000	$\underline{0.4227}$	0.1593	0.2870
Case 7	$\underline{0.1693}$	$\underline{0.1693}$	$\underline{0.3333}$	$\underline{0.3333}$	0.1168	0.1168
Case 8	$\underline{0.1530}$	0.1322	0.3333	$\underline{0.2929}$	0.1098	0.2027
Case 9	$\underline{0.1215}$	$\underline{0.1215}$	$\underline{0.2929}$	$\underline{0.2929}$	0.1761	0.1761
Case 10	$\underline{0.2356}$	0.2030	0.5774	$\underline{0.5000}$	0.2225	0.3926
Case 11	$\underline{0.2783}$	0.2560	0.6667	$\underline{0.4227}$	0.0559	0.3458
Case 12	0.2362	$\underline{0.2510}$	0.5000	$\underline{0.3333}$	0.0417	0.0494

Table 3: Support range and value function in the noisy-type Teraoka-Baston/Garnaev game

	b_1	b_2	m_1	m_2	$K_1(b)$	$K_2(b)$
Case 2	$\underline{0.3849}$	0.3333	0.6667	$\underline{0.5000}$	0.0911	0.2368
Case 4	$\underline{0.4196}$	0.3849	0.7071	$\underline{0.5774}$	0.1451	0.3457
Case 6	0.2000	$\underline{0.2157}$	0.5000	$\underline{0.4227}$	0.1692	0.3019
Case 8	0.1481	$\underline{0.1624}$	0.3333	$\underline{0.2929}$	0.1140	0.2094
Case 10	0.2373	$\underline{0.2568}$	0.5774	$\underline{0.5000}$	0.2398	0.4181
Case 11	$\underline{0.3179}$	0.2906	0.6667	$\underline{0.4227}$	0.0689	0.3647
Case 12	$\underline{0.2985}$	0.2435	0.5000	$\underline{0.3333}$	0.0438	0.1469

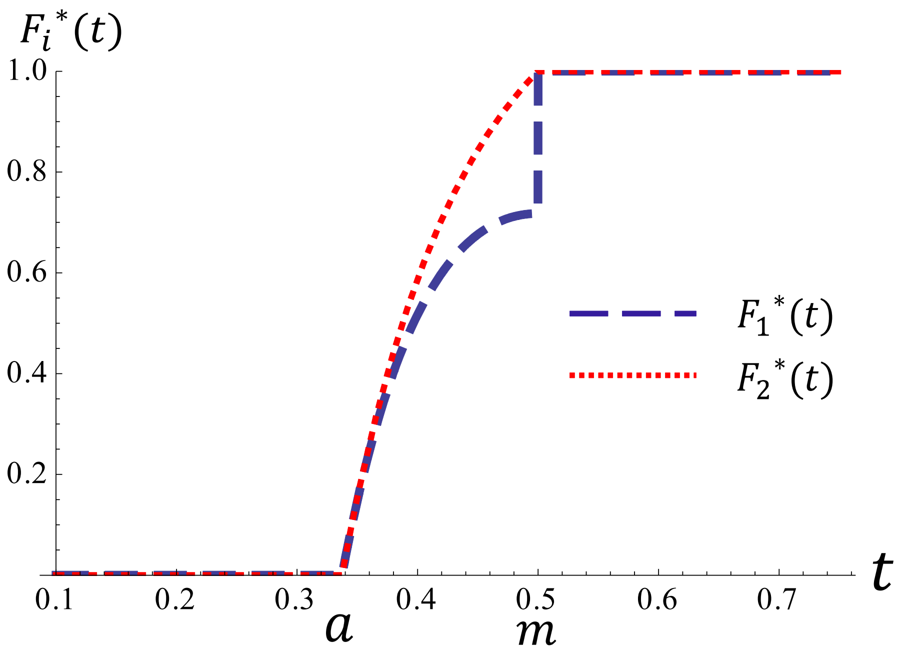

Figure 1: Behavior of Nash equilibrium strategy in the silent-type Teraoka-Baston/Garnaev game in Case 2

From this figure, we observe that only one player takes the Nash equilibrium strategy which is a mixed strategy in the noisy-type Teraoka-Baston/Garnaev game. Furthermore, it is seen that the player who takes a mixed strategy is able to select the support of their own strategy freely.

Next we compare the Nash equilibrium strategy in the silent/noisy-type Teraoka-Baston/Garnaev game with the Stackelberg strategy. In Case A and Case B, the model parameters are given by $A_1(t) = t, A_2(t) = t^2, H(t) = t$ and $A_1(t) = t^3, A_2(t) = t, H(t) = -(t-1)^2 + 1$, respectively. Suppose that Player 1 as a silent player and Player 2 as a noisy player exist in the silent/noisy game, and that Player 1 as a follower and Player 2 as a leader exist in the Stackelberg game. Table 4 presents the value function and the optimal payoff in two games. From this table, we can see that Player 2 of the Stackelberg game can get more expected profit than the value function in the silent/noisy game in both cases. Since the condition $b_1 \geq m_2$ holds in Case B, it is seen that the strategies of two games are equivalent, as shown in Remark 4.1, and that each player obtains the same value of the expected profit in the both games. Figures 3 and 4 show the behavior of Nash equilibrium strategy and Stackelberg

Table 4: Parameters and value functions in two games

			silent/noisy game		Stackelberg game	
	b_1	m_2	$M_1((F_1^*, m_1), F_2^*)$	$M_2((F_1^*, m_1), F_2^*)$	$M_1(x^*, y^*)$	$M_2(x^*, y^*)$
Case A	0.333	0.667	0.224	0.075	0.222	0.148
Case B	0.383	0.333	0.023	0.148	0.023	0.148

strategy in both cases, respectively. From Figure 3, we observe that both players of the silent/noisy-type game have to act earlier than the Stackelberg game under the condition of $b_1 < m_2$. From Figure 4, it can be seen that the strategy of the silent player is equivalent

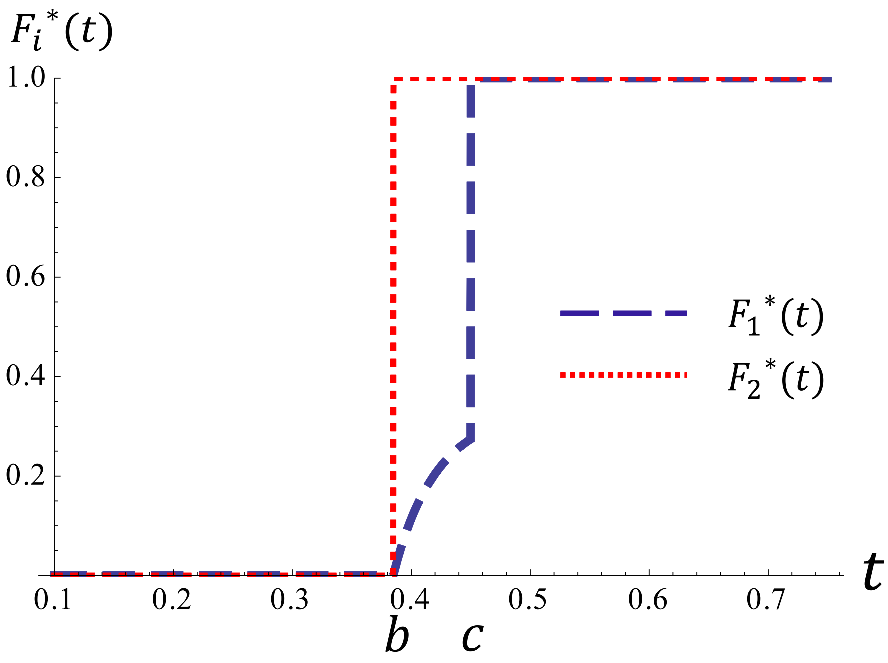

Figure 2: Behavior of Nash equilibrium strategy in the noisy-type Teraoka-Baston/Garnaev game in Case 2

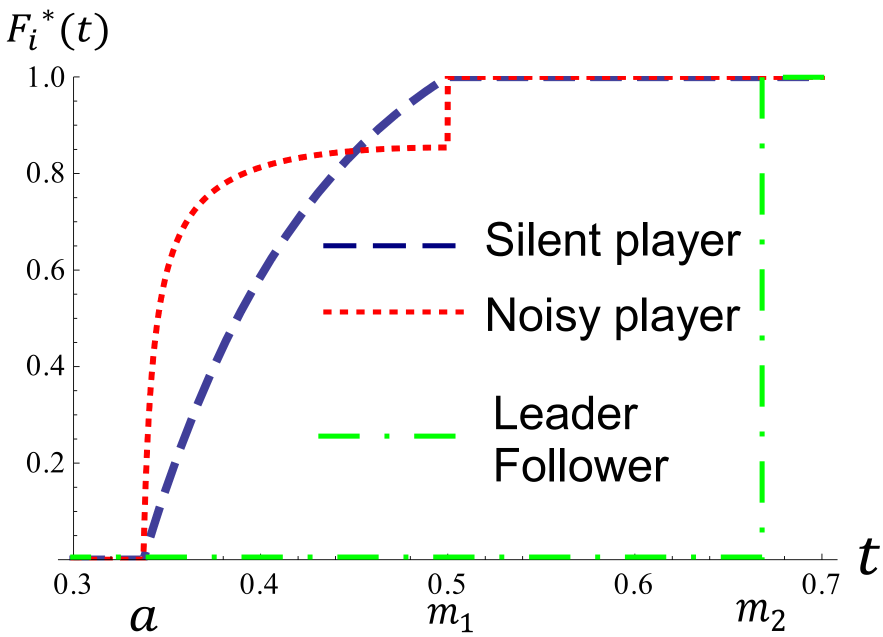

Figure 3: Comparison of Nash equilibrium strategy and Stackelberg strategy in Case A

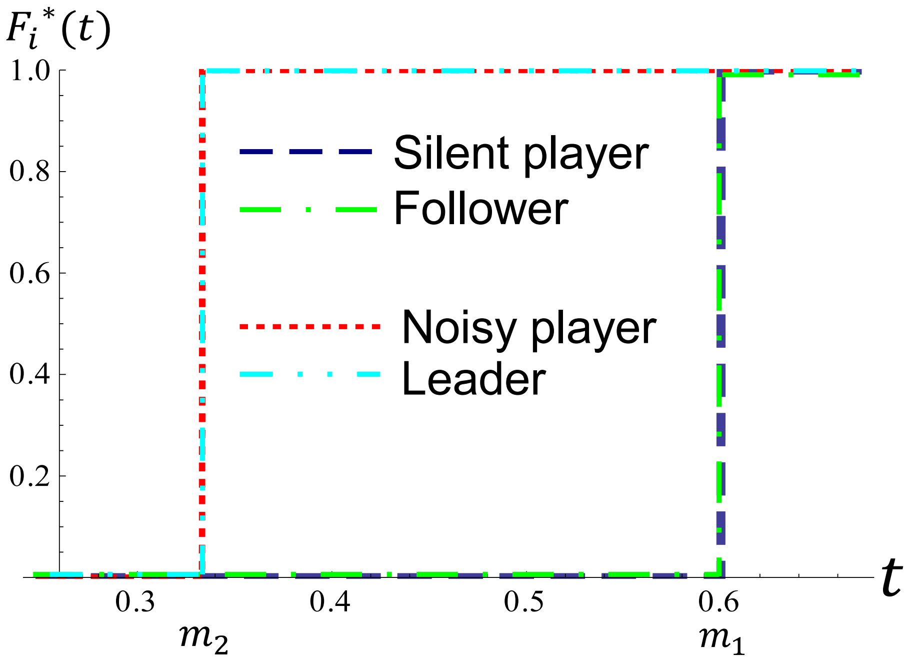

Figure 4: Comparison of Nash equilibrium strategy and Stackelberg strategy in Case B

to that of the follower, and that the strategy of the noisy player is equivalent to that of the leader under the condition of $b_1 \geq m_2$.

6. Conclusion

In this paper, we have formulated a unified marksmanship contest game with random termination and shown that our formulation split into five different marksmanship contest games. The first four games can be characterized as two-person nonzero-sum games of timing. The last game is the Stackelberg game with two players. The solutions of respective marksmanship contest games have been derived analytically under some conditions. Through simple numerical examples, we have confirmed the detailed behaviors of solutions which were the Nash equilibrium strategies or the Stackelberg strategies.

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