Journal of the Operations Research Society of Japan
Vol. 61, No. 2, April 2018, pp. 236–238

Corrigendum: "ERROR BOUNDS FOR LAST-COLUMN-BLOCK-AUGMENTED TRUNCATIONS OF BLOCK-STRUCTURED MARKOV CHAINS" Vol. 60, No. 3, 2017, pp. 271–320

Hiroyuki Masuyama
Kyoto University

(Received November 9, 2017)

Section 2.2 of Masuyama [2] presents a computable and nontrivial lower bound $\overline{\phi}_{K,N}^{(\beta)}$ for the factor $\overline{\phi}_{K}^{(\beta)}$ of the error bounds given in Theorems 2.1, 2.2 and 2.4. The author stated that the lower bound $\overline{\phi}_{K,N}^{(\beta)}$ exists because (see [2, Equation (2.66)])

$$\lim_{N\to\infty} \uparrow \overline{\phi}_{K,N}^{(\beta)} = \overline{\phi}_{K}^{(\beta)}, \tag{1}$$

where the symbol $\uparrow$ represents *"convergence from below"*. However, the proof of (1), presented in [2], is not complete. Thus, this corrigendum presents a complete proof of (1).

It follows from [1, Section 2.2, Proposition 2.14] that, for all $t \ge 0$ and $(k,i;\ell,j) \in \mathbb{F}^2$,

$$\lim_{N\to\infty} \uparrow [\exp\{\boldsymbol{Q}_{\mathbb{F}_N} t\}]_{(k,i;\ell,j)} = p^{(t)}(k,i;\ell,j),$$

where $[\exp\{\boldsymbol{Q}_{\mathbb{F}_N} t\}]_{(k,i;\ell,j)}$ denotes the $(k,i;\ell,j)$th element of $\exp\{\boldsymbol{Q}_{\mathbb{F}_N} t\}$. Therefore, by the monotone convergence theorem, we have, for all $(k,i;\ell,j) \in \mathbb{F}^2$,

$$\lim_{N\to\infty} \uparrow \int_0^\infty \beta e^{-\beta t} [\exp\{\boldsymbol{Q}_{\mathbb{F}_N} t\}]_{(k,i;\ell,j)} \,dt = \int_0^\infty \beta e^{-\beta t} p^{(t)}(k,i;\ell,j) dt > 0. \tag{2}$$

Using [2, Equations (2.3) and (2.59)], we rewrite (2) as

$$\lim_{N\to\infty} \uparrow \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) = \phi^{(\beta)}(k,i;\ell,j) > 0, \qquad \forall (k,i;\ell,j) \in \mathbb{F}^2. \tag{3}$$

Although $\phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j)$ is defined for $(k,i;\ell,j) \in (\mathbb{F}_N)^2$ (see [2, Equation (2.59)]), we set

$$\phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) = 0, \qquad (k,i) \in \mathbb{F} \setminus \mathbb{F}_N \text{ or } (\ell,j) \in \mathbb{F} \setminus \mathbb{F}_N. \tag{4}$$

It then follows from (3) and [2, Equation (2.65)] that $\{\overline{\phi}_{K,N}^{(\beta)}; N = K, K+1, \dots\}$ is nondecreasing and thus

$$\begin{aligned} \lim_{N\to\infty} \overline{\phi}_{K,N}^{(\beta)} &= \sup_{N\ge K} \overline{\phi}_{K,N}^{(\beta)} \\ &= \sup_{N\ge K} \sup_{(\ell,j)\in\mathbb{F}_N} \min_{(k,i)\in\mathbb{F}_K} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) \\ &= \sup_{N\ge K} \sup_{(\ell,j)\in\mathbb{F}} \min_{(k,i)\in\mathbb{F}_K} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j), \end{aligned} \tag{5}$$

where the last equality holds due to (4). Note here that the order of double supremum is interchangeable (see the lemma below), i.e.,

$$\sup_{N \geq K} \sup_{(\ell,j)\in\mathbb{F}} \min_{(k,i)\in\mathbb{F}_K} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) = \sup_{(\ell,j)\in\mathbb{F}} \sup_{N \geq K} \min_{(k,i)\in\mathbb{F}_K} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j). \tag{6}$$

Substituting (6) into (5), and using (3), we obtain

$$\begin{aligned}
\lim_{N\to\infty} \overline{\phi}_{K,N}^{(\beta)} &= \sup_{(\ell,j)\in\mathbb{F}} \sup_{N \geq K} \min_{(k,i)\in\mathbb{F}_K} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) \\
&= \sup_{(\ell,j)\in\mathbb{F}} \lim_{N\to\infty} \min_{(k,i)\in\mathbb{F}_K} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) \\
&= \sup_{(\ell,j)\in\mathbb{F}} \min_{(k,i)\in\mathbb{F}_K} \lim_{N\to\infty} \phi_{\mathbb{F}_N}^{(\beta)}(k,i;\ell,j) \\
&= \sup_{(\ell,j)\in\mathbb{F}} \min_{(k,i)\in\mathbb{F}_K} \phi^{(\beta)}(k,i;\ell,j) \\
&= \overline{\phi}_K^{(\beta)},
\end{aligned}$$

where the last equality follows from [2, Equation (2.10)]. As a result, we have proved that (1) holds.

We close this corrigendum by providing the lemma, which enables us to interchange the order of double supremum.

Lemma (Interchanging the Order of Double Supremum) *Let* $\{a_{n,m}; n, m \in \mathbb{N}\}$ *denote a sequence of real numbers, where* $\mathbb{N} = \{1, 2, 3, \dots\}$*. We then have*

$$\sup_{(n,m)\in\mathbb{N}^2} a_{n,m} = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m} = \sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} a_{n,m}.$$

Proof. By symmetry, it suffices to prove that

$$\sup_{(n,m)\in\mathbb{N}^2} a_{n,m} = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m}. \tag{7}$$

If

$$\sup_{(n,m)\in\mathbb{N}^2} a_{n,m} > \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m},$$

then, for some $(n', m') \in \mathbb{N}^2$, we have $a_{n',m'} > \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m}$ whereas, by definition, $a_{n',m'} \leq \sup_{m\in\mathbb{N}} a_{n',m} \leq \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m}$, which yields a contradiction. On the other hand, if

$$\sup_{(n,m)\in\mathbb{N}^2} a_{n,m} < \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m},$$

then

$$\begin{aligned}
\sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} a_{i,j} &\leq \sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} \sup_{(n,m)\in\mathbb{N}^2} a_{n,m} \\
&= \sup_{(n,m)\in\mathbb{N}^2} a_{n,m} < \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} a_{n,m},
\end{aligned}$$

which also yields a contradiction. Consequently, (7) holds. □

References

[1] W.J. Anderson: *Continuous-Time Markov Chains: An Applications-Oriented Approach* (Springer, New York, 1991).

[2] H. Masuyama: Error bounds for last-column-block-augmented truncations of block-structured Markov chains. *Journal of the Operations Research Society of Japan*, **60** (2017), 271–320.

Hiroyuki Masuyama
Graduate School of Informatics
Kyoto University
Kyoto 606-8501, Japan
E-mail: masuyama@sys.i.kyoto-u.ac.jp