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LINEAR REBALANCING STRATEGY FOR MULTI-PERIOD DYNAMIC PORTFOLIO OPTIMIZATION UNDER REGIME SWITCHES

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Abstract Although there is a growing interest of applying regime switching models to portfolio optimization, it has never been quite easy as yet to obtain analytical solutions under practical conditions such as self-financing constraints and short sales constraints. In this paper, we extend the linear rebalancing rule proposed in Moallemi and Sağlam [17] to regime switching models and provide a multi-period dynamic investment strategy that is comprised of a linear combination of factors with regime dependent coefficients. Under plausible mathematical assumptions, the problem to determine optimal coefficients maximizing a mean-variance utility penalized for transaction costs subject to self-financing and short sales constraints can be formulated as a quadratic programming which is easily solved numerically. To suppress an exponential increase of a number of optimization variables caused by regime switches, we propose a sample space reduction method. From numerical experiments under a practical setting, we confirm that our approach achieves sufficiently reasonable performances, even when sample space reduction is applied for longer investment horizon. The results also show superior performance of our approach to that of the optimal strategy without concerning transaction costs.

Keywords: Finance, multi-period dynamic portfolio optimization, regime switch, short sales constraint, self-financing constraint, linear rebalancing strategy

1. Introduction

Across all investment layers ranging from asset allocation to individual portfolio selections, quantitative models predict expected returns and variability. For those investors who are capable of appropriately specifying the model that grasps statistical nature of return processes observed in the market, growing number of literature in finance have attracted full attentions of them to establish portfolio optimization models that well describe actual investment circumstances.

Since the pioneering work of Markowitz [16], the mean-variance approach has been a fundamental model of portfolio optimizations and has been extended to many directions. One of the important extensions is a multi-period dynamic optimization. For example, Li and Ng [14] derives an analytical form of optimal portfolios under a self-financing condition. Related models to Li and Ng [14] are also investigated in Leippold et al. [13], Costa and Nabholz [4], and others. For an infinite horizon problem, Gârleanu and Pedersen [9] derives a closed form optimal portfolios for the mean-variance utility penalized for quadratic transaction costs. Extensions to impose practical investment constraints have also been discussed. Li, Zhou and Lim [15] derives a dynamic optimal portfolio for the mean-variance utility subject to a short sales constraint. Hibiki [11] proposes a hybrid simulation/tree approach

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for asset allocation with a budget constraint and confirms superior performance to the scenario tree model, see also related references therein. Takano and Gotoh [19, 20] develops a nonlinear control policy using kernel method for portfolio optimization under a short sales constraint. In the absence of transaction costs, Cui, Gao, Li and Li [5] achieves multi-period portfolio solutions for the mean-variance investment utility under a short sales constraint. Gao, Xiong and Li [8] solves the dynamic mean-variance-CVaR for a self-financing portfolio process.

On the other hand, there is a growing interest in applying regime switching models to portfolio optimizations recently. Zhou and Yin [23] studies a continuous time and a multi-period version of the Markowitz's mean-variance portfolio selection with regime switching. Costa and Araujo [3] studies a multi-period mean-variance strategy with regime dependent parameters under VaR constraints. Wu and Li [21] analyzes a multi-period mean-variance portfolio with regime switching and a stochastic cash flows. Chen, Yang and Yin [2] augments Zhou and Yin [23] to include liability information to solve for an asset liability management under a continuous-time Markov regime-switching model. Chen and Yang [1] extends Chen, Yang and Yin [2] to a multi-period regime-switching model and obtains optimal investments with uncontrollable liability under the Markowitz's mean-variance portfolio selection problem. Shen and Siu [18] investigates an optimal asset allocation problem in a regime-switching financial market where a short rate is governed by a regime switching Vasicek model and stock prices by regime-switching Geometric Brownian motion. Dombrovskii and Obyedko [6] investigates a problem to minimize deviations from a benchmark subject to a borrowing constraint under regime switches. Komatsu and Makimoto [12] extends Gârleanu and Pedersen [9] to regime switching factors and return processes. Yao, Li and Li [22] studies a dynamic optimal multi-period asset allocation associated with uncontrollable liability where a stochastic interest rate is governed by the discrete time Vasicek model, showing a prospective of the model extensions into regime dependent space. Those regime switching models are useful as they flexibly handle discontinuous fluctuations of the return process over time. However, if practical conditions such as self-financing constraints, short sales constraints and transaction costs are imposed, it is so rare to obtain analytical solutions that we need to resort to numerical optimizations.

This paper is devoted to contemplate a dynamic investment problem where asset prices are regime dependent and investment constraints are imposed. Our special focus is on Moallemi and Sağlam [17] advocating the linear rebalancing rules that apply to a wide class of optimal investment problems. In the context of factor models to predict returns to assets to invest, the idea is based on Gârleanu and Pedersen [9] showing for an infinite horizon problem without constraints that the optimal portfolio is a linear combination of a current portfolio and factors. Moallemi and Sağlam [17] then proposes to optimize a utility function in a class of investment strategies with a same form as Gârleanu and Pedersen [9]. A notable feature of this approach is that an obtained portfolio is dynamic as it reflects up-to-date observation of factors. It is also advantageous that the optimization problem is reduced to a quadratic programming even when practical conditions such as short sales constraints and quadratic transaction costs are imposed.

In this paper, we extend the linear rebalancing rule to regime switching models. For an infinite horizon problem under regime switches without constraints, Komatsu and Makimoto [12] proves that an optimal portfolio has a same form as Gârleanu and Pedersen [9] while weight matrices between the current portfolio and factors are regime dependent. This leads to a dynamic investment strategy comprised of a linear combination of past and current factors with regime dependent coefficient matrices. Though this is a natural extension of Moallemi and Sağlam [17], a formulation of the optimization problem becomes much complicated due to the regime switches. Under plausible mathematical assumptions, we explicitly represent the optimization problem as a quadratic programming. Another difficulty caused by the regime switches is that a number of optimization variables increases exponentially fast as an investment horizon increases. To suppress the state space explosion, we propose a sample space reduction method under which the number of variables is a polynomial function of the investment horizon.

To check the usefulness of our approach for a realistic number of factors and assets, we conduct numerical experiments using the model with the parameters estimated from market data. From those numerical experiments, we confirm that our approach achieves sufficiently reasonable performances, even when sample space reduction is applied for longer investment horizon. The results also show superior performance of our approach to that of the optimal strategy without concerning transaction costs. By virtue of the optimization problems solved under practical conditions, the most significant contribution to investment practices enables a much broad spectrum in the vast majority of the investment society to implement regime dependent multi-period optimal portfolios.

This paper is constructed as follows. Section 2 sets up the portfolio optimization problem. We explain the linear rebalancing strategy under regime switches in Section 3. Section 4 is devoted to derive an explicit formulation of the optimization problem as a quadratic programming subject to second order cone constraints. In Section 5, we conduct the numerical experiments to confirm that the proposed approach works well enough under the practical setting of the model. Finally, Section 6 concludes the paper.

2. Portfolio Optimization Problem under Regime Switches

The portfolio optimization model considered in this paper is similar to that in Komatsu and Makimoto [12] which introduced regime switches into Gârleanu and Pedersen [9]. We consider an economy with N assets traded at $t = 1, 2, \ldots$. The excess return of asset n to the market return between t and $t+1$ is $r_n(t+1)$. We assume that an $N \times 1$ excess return vector $\boldsymbol{r}(t) = [r_1(t), \ldots, r_N(t)]^\top$ ($\top$ denotes transpose) is given by

$$\boldsymbol{r}(t+1) = \boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t) + \boldsymbol{u}_{I(t+1)}(t+1) \tag{2.1}$$

where $\{I(t)\}$ is a regime process on $\mathcal{J} = \{1, \ldots, J\}$ which represents discontinuous state changes of the market. The first term $\boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t)$ denotes the expected excess return known to the investor at t where $\boldsymbol{f}(t)$ is an $M \times 1$ vector of factors that an investor chooses to predict excess returns. $\boldsymbol{L}_{I(t+1)}$ is an $N \times M$ matrix of factor loadings such that $\boldsymbol{L}_{I(t+1)} = \boldsymbol{L}_i$ when $I(t+1) = i$. The second term $\boldsymbol{u}_{I(t+1)}(t+1)$ represents an unpredictable noise. We assume that, when $I(t+1) = i$, $\boldsymbol{u}_{I(t+1)}(t+1)$ follows a multivariate normal distribution with $\mathrm{E}\left(\boldsymbol{u}_{I(t+1)}(t+1) \middle| I(t+1) = i\right) = \boldsymbol{0}$ for all i where $\boldsymbol{0}$ denotes a zero vector, and a covariance matrix $\boldsymbol{W}_i = \mathrm{V}\left(\boldsymbol{u}_{I(t+1)}(t+1) \middle| I(t+1) = i\right)$.

The dynamics of the factor is modeled by a first order regime-switching vector autoregressive process

$$\boldsymbol{f}(t+1) = \boldsymbol{\mu}_{I(t+1)} + \boldsymbol{\Phi}_{I(t+1)}\boldsymbol{f}(t) + \boldsymbol{\epsilon}_{I(t+1)}(t+1). \tag{2.2}$$

$\boldsymbol{\mu}_{I(t+1)}$ is an $M \times 1$ vector determining the level of mean-reversion and $\boldsymbol{\Phi}_{I(t+1)}$ is an $M \times M$ coefficient matrix that are respectively given as $\boldsymbol{\mu}_{I(t+1)} = \boldsymbol{\mu}_i$ and $\boldsymbol{\Phi}_{I(t+1)} = \boldsymbol{\Phi}_i$ when $I(t+1) = i$. $\boldsymbol{\epsilon}_{I(t+1)}(t+1)$ is a vector of noise terms affecting the factors. As for $\boldsymbol{u}_{I(t+1)}(t+1)$, we assume that, when $I(t+1) = i$, $\boldsymbol{\epsilon}_{I(t+1)}(t+1)$ follows a multivariate normal distribution with $\mathrm{E}\left(\boldsymbol{\epsilon}_{I(t+1)}(t+1) \middle| I(t+1) = i\right) = \boldsymbol{0}$ for all i and a covariance matrix $\boldsymbol{\Sigma}_i =$

$\mathrm{V}\left(\boldsymbol{\epsilon}_{I(t+1)}(t+1)\middle|\, I(t+1)=i\right)$. We also assume that the factor process $\{\boldsymbol{f}(t)\}$ is stationary in time. Conditions for the stationarity of the regime-switching vector autoregressive process are given in Francq and Zakoaïan [7].

As is many existing literatures, we assume the regime process $\{I(t)\}$ follows an irreducible Markov chain on $\mathcal{J}$ the transition probability matrix of which is given by $\boldsymbol{P} = [p_{i,j}]$ with $p_{i,j} = \mathrm{P}\left(I(t+1)=j|\, I(t)=i\right)$. The noise terms $\boldsymbol{u}_{I(t)}(t)$ and $\boldsymbol{\epsilon}_{I(t)}(t)$ are assumed to be conditionally independent in the sense that, given any sample path of the regime process $I(1)=i_1, I(2)=i_2,\ldots$, $\boldsymbol{u}_{i_s}(s)$ and $\boldsymbol{\epsilon}_{i_t}(t)$ are independent of each other for all s and t.

Let $x_n(t)$ be an amount of investment to asset n at t and denote the portfolio by $\boldsymbol{x}(t) = [x_1(t),\ldots,x_N(t)]^\top$. We assume that quadratic transaction cost

$$TC = \frac{1}{2}\{\boldsymbol{x}(t)-\boldsymbol{x}(t-1)\}^\top \boldsymbol{B}_{I(t+1)}\{\boldsymbol{x}(t)-\boldsymbol{x}(t-1)\}$$

will be incurred for trading $\boldsymbol{x}(t)-\boldsymbol{x}(t-1)$ where $\boldsymbol{B}_i$ is a symmetric positive definite cost matrix when $I(t+1)=i$. Let $z(t)=\boldsymbol{x}(t)^\top\boldsymbol{1}$ be the amount of portfolio where $\boldsymbol{1}=[1,\ldots,1]^\top$ and let $\boldsymbol{w}(t)=\frac{\boldsymbol{x}(t)}{z(t)}$ denote a portfolio weight vector. From (2.1), the excess return between t and $t+1$ subject to transaction cost is

$$\begin{aligned} R(t+1) &= \frac{\boldsymbol{x}(t)^\top \boldsymbol{r}(t+1) - TC}{z(t)} \\ &= \boldsymbol{w}(t)^\top\{\boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t)+\boldsymbol{u}_{I(t+1)}(t+1)\} - \frac{1}{2}z(t)\Delta\boldsymbol{w}(t)^\top\boldsymbol{B}_{I(t+1)}\Delta\boldsymbol{w}(t)\end{aligned}$$

where $\Delta\boldsymbol{w}(t)=\boldsymbol{w}(t)-\frac{z(t-1)}{z(t)}\boldsymbol{w}(t-1)$. In a dynamic optimization framework, an investor determines $\boldsymbol{w}(t)$ to maximize a utility of $R(t+1)$ based on the information available at t. Let $\boldsymbol{f}[t]=\{\boldsymbol{f}(s), s\le t\}$, $\boldsymbol{r}[t]=\{\boldsymbol{r}(s), s\le t\}$ and $\boldsymbol{I}[t]=\{I(s), s\le t\}$ respectively be histories of the each process up to t and let $H[t]=\{\boldsymbol{f}[t],\boldsymbol{r}[t],\boldsymbol{I}[t]\}$. To represent the objective function explicitly, we assume that the investor observes $H[t]$, while $\boldsymbol{I}[t]$ is not directly observed from data in general. We further assume that the investor is able to predict $I(t+1)$ with certainty at t. Under these assumptions, the conditional mean-variance utility subject to transaction costs is given by

$$\begin{aligned} &\mathrm{E}\left(R(t+1)|\, H[t], I(t+1)\right) - \frac{\lambda}{2}\mathrm{V}\left(R(t+1)|\, H[t], I(t+1)\right) \\ &\quad = \boldsymbol{w}(t)^\top\boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t) - \frac{\lambda}{2}\boldsymbol{w}(t)^\top\boldsymbol{W}_{I(t+1)}\boldsymbol{w}(t) - \frac{1}{2}z(t)\Delta\boldsymbol{w}(t)^\top\boldsymbol{B}_{I(t+1)}\Delta\boldsymbol{w}(t) \qquad (2.3)\end{aligned}$$

where λ denotes the investor's coefficient of risk aversion.

In sum, the investor attempts to maximize the expected sum of the mean-variance utility penalized for transaction costs from current time $t=1$ until investment horizon T:

$$\sum_{t=1}^{T}\rho^{t-1}\{U_1(t)-\frac{\lambda}{2}U_2(t)-\frac{1}{2}U_3(t)\} \qquad (2.4)$$

where $\rho\in[0,1]$ is a discount rate and

$$\begin{aligned} U_1(t) &= \mathrm{E}\left(\boldsymbol{w}(t)^\top\boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t)\middle|\,\boldsymbol{x}(0), H[1]\right) \qquad (2.5)\\ U_2(t) &= \mathrm{E}\left(\boldsymbol{w}(t)^\top\boldsymbol{W}_{I(t+1)}\boldsymbol{w}(t)\middle|\,\boldsymbol{x}(0), H[1]\right) \qquad (2.6)\\ U_3(t) &= \mathrm{E}\left(z(t)\Delta\boldsymbol{w}(t)^\top\boldsymbol{B}_{I(t+1)}\Delta\boldsymbol{w}(t)\middle|\,\boldsymbol{x}(0), H[1]\right). \qquad (2.7)\end{aligned}$$

From the definition, $\boldsymbol{w}(t)$ satisfies a self-financing constraint $\boldsymbol{w}(t)^\top \mathbf{1} = 1$. We also assume that short sales are not allowed, i.e. $\boldsymbol{w}(t) \geq \mathbf{0}$. Among all investment decisions $\boldsymbol{w}(1), \ldots, \boldsymbol{w}(T)$, only $\boldsymbol{w}(1)$ is deterministic and $\boldsymbol{w}(t)$ for $t = 2, \ldots, T$ are stochastic since $\boldsymbol{w}(t)$ is determined based on $H[t]$ that are uncertain at $t = 1$. The expectations in (2.5)$\sim$(2.7) are thus calculated with respect to all possible sample paths of $H[t]$ for $t = 2, \ldots, T$.

Remark 2.1. The predictability of $I(t+1)$ is assumed to make conditional mean-variance utility computationally tractable. Without this assumption, mean and covariance of $R(t+1)$ conditioned only on $H[t]$ become

$$\mathrm{E}\left(R(t+1) | \, H[t] \right) = \boldsymbol{w}(t)^\top \sum_{j=1}^{J} p_{I(t),j} \boldsymbol{L}_j \boldsymbol{f}(t)$$

and $\mathrm{V}\left(R(t+1) | \, H[t] \right) = \mathrm{E}\left(R(t+1)^2 | \, H[t] \right) - \mathrm{E}\left(R(t+1) | \, H[t] \right)^2$ where

$$\mathrm{E}\left(R(t+1)^2 \middle| \, H[t] \right) = \boldsymbol{w}(t)^\top \left[\sum_{j=1}^{J} p_{I(t),j} \{ \boldsymbol{L}_j \boldsymbol{f}(t) \boldsymbol{f}(t)^\top \boldsymbol{L}_j^\top + \boldsymbol{W}_j \} \right] \boldsymbol{w}(t).$$

Since these expressions are so complicated than (2.3) that it is impossible to explicitly represent the objective function as in Proposition 4.1 for the linear rebalancing strategy. In Section 5.1, we will explain how to estimate and predict the regime process. The validity of the assumption is also discussed from the viewpoint of empirical studies.

3. Linear Rebalancing Strategy under Regime Switches

Our aim is to develop a multi-period dynamic investment strategy for the objective function (2.4) subject to self-financing and short sales constraints. In a dynamic optimization framework, a future investment decision $\boldsymbol{w}(t)$ is made based on the observations up to t. Since the number of possible sample paths of the regime process as well as factor process grows exponentially fast as the time horizon T gets longer, the problem becomes much involved compared with the static optimization.

To avoid the difficulty, Moallemi and Sağlam [17] proposed a linear rebalancing strategy (LRS for short) for multi-period dynamic portfolio optimization. The idea is based on Gârleanu and Pedersen [9] where they have proved that the optimal investment for an infinite horizon problem is given by

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t-1) + \boldsymbol{B}\boldsymbol{f}(t) + \boldsymbol{b} \tag{3.1}$$

for some matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and vector $\boldsymbol{b}$ determined from the model parameters. Starting with an initial portfolio $\boldsymbol{x}(0)$, iterative substitutions of (3.1) show that $\boldsymbol{x}(t)$ can be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{A}^t \boldsymbol{x}(0) + \boldsymbol{A}^{t-1} \boldsymbol{B} \boldsymbol{f}(1) + (\boldsymbol{A}^{t-1} + \cdots + \boldsymbol{A} + \boldsymbol{I}) \boldsymbol{b} + \sum_{s=2}^{t} \boldsymbol{A}^{t-s} \boldsymbol{B} \boldsymbol{f}(s) \tag{3.2}$$

where $\boldsymbol{I}$ is an identity matrix. Since (3.2) indicates that the optimal investment policy for the infinite horizon problem without regime switches is a linear combination of past and current factors, Moallemi and Sağlam [17] suggests a dynamic investment strategy of the form

$$\boldsymbol{x}(t) = \boldsymbol{C}_t(1) + \sum_{s=2}^{t} \boldsymbol{C}_t(s) \boldsymbol{f}(s)$$

where a summation $\sum_{s=a}^{b}$ should be understood as 0 for $a > b$. The coefficient matrices $\boldsymbol{C}_t(s)$'s are determined so as to optimize a given objective function for a finite horizon problem.

Under the regime switching circumstance, Komatsu and Makimoto [12] extends Gârleanu and Pedersen [9] in such a way that the optimal investment is shown to be

$$\boldsymbol{x}(t) = \boldsymbol{A}_i \boldsymbol{x}(t-1) + \boldsymbol{B}_i \boldsymbol{f}(t) + \boldsymbol{b}_i \tag{3.3}$$

when $I(t) = i$. Since (3.3) is a natural extension of (3.1), we apply the idea of the LRS to the case with multiple regimes in a following way. Suppose that an investor is going to make investment decisions $\boldsymbol{w}(1), \ldots, \boldsymbol{w}(T)$ at $t = 1$.

LRS under regime switches :

- For $t = 1, \ldots, T$, the investment decision is given by

$$\boldsymbol{w}(t) = \boldsymbol{C}_{\boldsymbol{I}[t]}(1) + \sum_{s=2}^{t} \boldsymbol{C}_{\boldsymbol{I}[t]}(s) \boldsymbol{f}(s) \tag{3.4}$$

  where $\boldsymbol{C}_{\boldsymbol{I}[t]}(1)$ is an $N \times 1$ vector and $\boldsymbol{C}_{\boldsymbol{I}[t]}(s)$ $(s = 2, \ldots, t)$ is an $N \times M$ matrix depending on a sample path $\boldsymbol{I}[t] = \{I(1), \ldots, I(t)\}$ of the regime process.
- For all $t = 1, \ldots, T$ and all possible sample path $\boldsymbol{I}[t]$, coefficient matrices $\boldsymbol{C}_{\boldsymbol{I}[t]}(s)$ $(s = 1, \ldots, t)$ are determined so as to maximize the objective function (2.4), see Section 4 for details.

We will give a simple example to illustrate LRS. It is noted that, since $\boldsymbol{C}_{\boldsymbol{I}[t]}(1), \ldots, \boldsymbol{C}_{\boldsymbol{I}[t]}(t)$ are defined for each $\boldsymbol{I}[t]$, $\boldsymbol{C}_{\boldsymbol{I}[s]}(r) \neq \boldsymbol{C}_{\boldsymbol{I}[t]}(r)$ for $s \neq t$.

Example 3.1. Suppose $\boldsymbol{f}(1)$ and $I(1) = 1$ realize at $t = 1$. Given $\boldsymbol{f}(1)$ and $I(1) = 1$, an investor solves an optimization problem in Section 4 to compute $\boldsymbol{C}_{\boldsymbol{I}[t]}(s)$ for all $t = 1, \ldots, T$, $\boldsymbol{I}[t] = \{1, I(2), \ldots, I(t)\}$ and $s = 1, \ldots, t$. Since $I(1) = 1$, $\boldsymbol{w}(1) = \boldsymbol{C}_1(1)$ is chosen at $t = 1$. If $\boldsymbol{f}(2)$ and $I(2) = 1$ realize at $t = 2$, the investment decision is $\boldsymbol{w}(2) = \boldsymbol{C}_{1,1}(1) + \boldsymbol{C}_{1,1}(2)\boldsymbol{f}(2)$. And if $\boldsymbol{f}(3)$ and $I(3) = 2$ follow at $t = 3$, then $\boldsymbol{w}(3) = \boldsymbol{C}_{1,1,2}(1) + \boldsymbol{C}_{1,1,2}(2)\boldsymbol{f}(2) + \boldsymbol{C}_{1,1,2}(3)\boldsymbol{f}(3)$. Investment decisions proceed in a similar way until $t = T$.

Hereafter, we denote the investment decision by $\boldsymbol{w}_{\boldsymbol{I}[t]}$ rather than $\boldsymbol{w}(t)$ to clarify the dependency of $\boldsymbol{w}(t)$ on $\boldsymbol{I}[t]$ under LRS. Since $\boldsymbol{w}_{\boldsymbol{I}[t]}$ is given as a linear combination of $\boldsymbol{f}(2), \ldots, \boldsymbol{f}(t)$ whose coefficient matrices $\boldsymbol{C}_{\boldsymbol{I}[t]}(2), \ldots, \boldsymbol{C}_{\boldsymbol{I}[t]}(t)$ are selected according to the regime process, LRS is a dynamic strategy. Only $\boldsymbol{w}_{\boldsymbol{I}[1]}$ is given deterministically.

Since $\boldsymbol{C}_{\boldsymbol{I}[t]}(1)$ is $N \times 1$ and $\boldsymbol{C}_{\boldsymbol{I}[t]}(s)$ $(s = 2, \ldots, t)$ is $N \times M$, and there are J^{t-1} possibilities for $\boldsymbol{I}[t]$ given $I(1)$, the total number of variables in the optimization is

$$\sum_{t=1}^{T} (1 + (t-1)M) N J^{t-1}. \tag{3.5}$$

As the investment horizon T gets longer, the number of variables increases exponentially fast. In Section 5.3, we will propose effective variable reduction method to suppress the state space explosion.

4. Explicit Formulation of the Optimization Problem for LRS

In this section, we give an explicit representation of the optimization problem to compute $\boldsymbol{C}_{\boldsymbol{I}[t]}(s)$ in (3.4) which maximizes (2.4). By introducing regime switches, it becomes much involved to express the objective function (2.4) explicitly in terms of $\boldsymbol{C}_{\boldsymbol{I}[t]}(s)$ compared with Moallemi and Sağlam [17] for LRS without regime switches. In addition to future uncertainty of the factor process, we need to take regime switches into account.

We define

$$\boldsymbol{C}_{\boldsymbol{I}[t]} = \left[\boldsymbol{C}_{\boldsymbol{I}[t]}(1), \boldsymbol{C}_{\boldsymbol{I}[t]}(2), \ldots, \boldsymbol{C}_{\boldsymbol{I}[t]}(t)\right], \quad t = 1, \ldots, T$$

and

$$\boldsymbol{F}[1] = 1, \qquad \boldsymbol{F}[t] = \begin{bmatrix} 1 \\ \boldsymbol{f}(2) \\ \vdots \\ \boldsymbol{f}(t) \end{bmatrix}, \quad t = 2, \ldots, T$$

to represent $\boldsymbol{w}_{\boldsymbol{I}[t]} = \boldsymbol{C}_{\boldsymbol{I}[t]}\boldsymbol{F}[t]$. As shown in Proposition 4.1 below, $U_1(t)$ and $U_2(t)$ can be represented as linear and quadratic forms in terms of $\boldsymbol{C}_{\boldsymbol{I}[t]}$ while $U_3(t)$ is not since $z(t)$ entailed in (2.7) depends on the investment strategy. To obtain an explicit expression, we approximate the amount of the portfolio $z(t)$ under LRS by the expected amount of the portfolio obtained by iteratively solving single-period mean-variance optimization problems subject to transaction costs and self-financing/short sales constraints. We denote the approximated value of $z(t)$ by $\xi_{\boldsymbol{I}[t]}$ since it depends on $\boldsymbol{I}[t]$. To keep the clarity of presentation, the computational procedure of $\xi_{\boldsymbol{I}[t]}$ is explained in Appendix A.1.

In what follows, we will use the following notations. For an $a \times b$ matrix $\boldsymbol{G} = [g_{i,j}]$, we define an $ab \times 1$ vector by $\mathrm{vec}(\boldsymbol{G}) = (g_{1,1}, \ldots, g_{a,1}, \ldots, g_{1,b}, \ldots, g_{a,b})^\top$. $\boldsymbol{I}_N$ is an N dimensional identity matrix and $\otimes$ denotes the Kronecker product of vectors/matrices. From Markovian property of the regime process, the probability of a sample path $\boldsymbol{I}[t] = \{I(1), \ldots, I(t)\}$ conditioned on $I(1)$ is given by $p(\boldsymbol{I}[t]) = \prod_{s=2}^{t} p_{I(s-1),I(s)}$. We also denote by $\sum_{\boldsymbol{I}[t]}$ a summation over all J^{t-1} sample paths of $\boldsymbol{I}[t]$ starting with $I(1)$. The expectation conditioned on $\boldsymbol{f}(1)$ and $\boldsymbol{I}[t]$ is expressed as $\mathrm{E}_t^*(\ ) = \mathrm{E}(\ \mid \boldsymbol{f}(1), \boldsymbol{I}[t])$.

Proposition 4.1. Given $\boldsymbol{f}(1)$ and $I(1)$, (2.5) and (2.6) are expressed as

$$U_1(t) = \begin{cases} \boldsymbol{C}_{\boldsymbol{I}[1]}^\top \boldsymbol{L}_{I(2)} \boldsymbol{f}(1), & t = 1 \\ \displaystyle\sum_{\boldsymbol{I}[t]} p(\boldsymbol{I}[t]) \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top)^\top \{\boldsymbol{I}_N \otimes \mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{f}(t)^\top\right)\} \mathrm{vec}(\boldsymbol{L}_{I(t+1)}^\top), & t = 2, \ldots, T \end{cases} \tag{4.1}$$

$$U_2(t) = \begin{cases} \boldsymbol{C}_{\boldsymbol{I}[1]}^\top \boldsymbol{W}_{I(2)} \boldsymbol{C}_{\boldsymbol{I}[1]}, & t = 1 \\ \displaystyle\sum_{\boldsymbol{I}[t]} p(\boldsymbol{I}[t]) \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top)^\top \{\boldsymbol{W}_{I(t+1)} \otimes \mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{F}[t]^\top\right)\} \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top), & t = 2, \ldots, T. \end{cases} \tag{4.2}$$

When $z(t)$ is approximated by $\xi_{\boldsymbol{I}[t]}$, (2.7) is expressed as

$$U_3(t) = \begin{cases} \xi_{\boldsymbol{I}[1]} \left\{\boldsymbol{C}_{\boldsymbol{I}[1]} - \dfrac{\xi_{\boldsymbol{I}[0]}}{\xi_{\boldsymbol{I}[1]}} \boldsymbol{w}(0)\right\}^\top \boldsymbol{B}_{I(2)} \left\{\boldsymbol{C}_{\boldsymbol{I}[1]} - \dfrac{\xi_{\boldsymbol{I}[0]}}{\xi_{\boldsymbol{I}[1]}} \boldsymbol{w}(0)\right\}, & t = 1 \\ \displaystyle\sum_{\boldsymbol{I}[t]} p(\boldsymbol{I}[t]) \xi_{\boldsymbol{I}[t]} \mathrm{vec}(\Delta \boldsymbol{C}_{\boldsymbol{I}[t]}^\top)^\top \{\boldsymbol{B}_{I(t+1)} \otimes \mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{F}[t]^\top\right)\} \mathrm{vec}(\Delta \boldsymbol{C}_{\boldsymbol{I}[t]}^\top), & t = 2, \ldots, T \end{cases} \tag{4.3}$$

where $\Delta \boldsymbol{C}_{\boldsymbol{I}[t]} = \boldsymbol{C}_{\boldsymbol{I}[t]} - [\frac{\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}} \boldsymbol{C}_{\boldsymbol{I}[t-1]}, \boldsymbol{O}_{N,M}]$ with $\boldsymbol{O}_{N,M}$ being an $N \times M$ zero matrix.

Proof. See Appendix A.2. □

The conditional expectations appeared in (4.1)∼(4.3) can be explicitly represented by the model parameters as shown in Proposition 4.3 at the end of this section. The summand in (4.3) is expanded as

$$\begin{aligned}
&\mathrm{vec}(\Delta \boldsymbol{C}_{\boldsymbol{I}[t]}^\top)^\top \left\{\boldsymbol{B}_{I(t+1)} \otimes \mathrm{E}_t^* \left(\boldsymbol{F}[t]\boldsymbol{F}[t]^\top\right)\right\} \mathrm{vec}(\Delta \boldsymbol{C}_{\boldsymbol{I}[t]}^\top) \\
&= \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top)^\top \left\{\boldsymbol{B}_{I(t+1)} \otimes \mathrm{E}_t^* \left(\boldsymbol{F}[t]\boldsymbol{F}[t]^\top\right)\right\} \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top) \\
&\quad + \frac{\xi_{\boldsymbol{I}[t-1]}^2}{\xi_{\boldsymbol{I}[t]}^2} \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t-1]}^\top)^\top \left\{\boldsymbol{B}_{I(t+1)} \otimes \mathrm{E}_t^* \left(\boldsymbol{F}[t-1]\boldsymbol{F}[t-1]^\top\right)\right\} \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t-1]}^\top) \\
&\quad - \frac{2\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}} \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t-1]}^\top)^\top \left\{\boldsymbol{B}_{I(t+1)} \otimes \mathrm{E}_t^* \left(\boldsymbol{F}[t-1]\boldsymbol{F}[t]^\top\right)\right\} \mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top).
\end{aligned}$$

This together with (4.1) and (4.2) imply that the objective function (2.4) is expressed as a sum of the linear and quadratic forms of $\mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top)$. Though (4.3) is an approximation of the transaction costs, we note that the level of transaction costs is generally lower than that of expected returns. Concerning that $U_1(t)$ and $U_2(t)$ are exactly represented in Proposition 4.1, we expect that it does not have a strong impact on the optimal LRS to approximate $z(t)$ by $\xi_{\boldsymbol{I}[t]}$.

We next consider the self-financing constraint $\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1} = 1$ and the short sales constraint $\boldsymbol{w}_{\boldsymbol{I}[t]} \geq \mathbf{0}$. Recall that $\boldsymbol{w}_{\boldsymbol{I}[1]}$ is deterministic while $\boldsymbol{w}_{\boldsymbol{I}[t]}$ for $t = 2, \ldots, T$ are stochastic under LRS. For $t = 1$, these constraints become linear constraints

$$\boldsymbol{w}_{\boldsymbol{I}[1]}^\top \mathbf{1} = \boldsymbol{C}_{\boldsymbol{I}[1]}^\top \mathbf{1} = 1 \tag{4.4}$$

$$\boldsymbol{w}_{\boldsymbol{I}[1]} = \boldsymbol{C}_{\boldsymbol{I}[1]} \geq \mathbf{0} \tag{4.5}$$

and are easily incorporated into optimizations. On the other hand, it is impossible for the LRS to satisfy self-financing/short sales constraints with certainty for $t \geq 2$. We therefore impose the following stochastic constraints

$$\mathrm{P}\left(|\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1} - 1| > \delta\right) \leq p_b, \qquad \forall \boldsymbol{I}[t], \quad t = 2, \ldots, T \tag{4.6}$$

$$\mathrm{P}\left(w_{\boldsymbol{I}[t],n} < 0\right) \leq p_s, \qquad n = 1, \ldots, N, \quad \forall \boldsymbol{I}[t], \quad t = 2, \ldots, T \tag{4.7}$$

where $w_{\boldsymbol{I}[t],n}$ denotes an n-th element of $\boldsymbol{w}_{\boldsymbol{I}[t]}$. Parameters δ, p_b and p_s are chosen by an investor to control how strictly the self-financing/short sales constraints are satisfied. We set $\delta = 0.025$ and $p_b = p_s = 0.05$ for the numerical experiments in Section 5. Since $\boldsymbol{w}_{\boldsymbol{I}[t]}$ follows a multivariate normal distribution given $\boldsymbol{I}[t]$, these constraints can be transformed into second order cone constraints.

Proposition 4.2. For $t = 2, \ldots, T$, the self-financing constraint (4.6) is represented by

$$\mathbf{1}^\top \boldsymbol{C}_{\boldsymbol{I}[t]} \mathrm{E}_t^* \left(\boldsymbol{F}[t]\right) = 1, \qquad \forall \boldsymbol{I}[t], \quad t = 2, \ldots, T \tag{4.8}$$

$$\|\mathbf{1}^\top \boldsymbol{C}_{\boldsymbol{I}[t]} \boldsymbol{\Theta}_{\boldsymbol{I}[t]}^\top\|_2 \leq \frac{\delta}{\Phi^{-1}(1 - p_b/2)}, \qquad \forall \boldsymbol{I}[t], \quad t = 2, \ldots, T \tag{4.9}$$

and the short sales constraint (4.7) becomes

$$\boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \mathrm{E}_t^* \left(\boldsymbol{F}[t]\right) \geq \Phi^{-1}(1 - p_s) \|\boldsymbol{\Theta}_{\boldsymbol{I}[t]} \boldsymbol{c}_{\boldsymbol{I}[t],n}\|_2, \qquad n = 1, \ldots, N, \quad \forall \boldsymbol{I}[t], \quad t = 2, \ldots, T \tag{4.10}$$

for $p_s \in (0, 0.5)$ where $\Phi^{-1}(\ )$ is an inverse cumulative standard normal distribution function, $\|\ \|_2$ is Euclidean norm, $\boldsymbol{c}_{\boldsymbol{I}[t],n}^\top$ denotes an n-th row vector of $\boldsymbol{C}_{\boldsymbol{I}[t]}$, and the matrix $\boldsymbol{\Theta}_{\boldsymbol{I}[t]}$ is given by (A.5) in Appendix A.2.

Proof. See Appendix A.2. □

From Propositions 4.1 and 4.2, the dynamic portfolio optimization problem for LRS under consideration is formulated as a quadratic programming subject to nonnegative and second order cone constraints:

$$\begin{array}{ll} \text{maximize} & (2.4) \text{ with } (4.1)\sim(4.3) \\ \text{subject to} & (4.4), (4.5), (4.8), (4.9) \text{ and } (4.10) \end{array} \tag{4.11}$$

Once the optimization problem is solved and $\boldsymbol{C}_{\boldsymbol{I}[t]}$ for $t = 1, \ldots, T$ are computed, the portfolio $\boldsymbol{x}_{\boldsymbol{I}[t]}$ is given by

$$\boldsymbol{x}_{\boldsymbol{I}[t]} = \frac{z(t)}{\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1}} \boldsymbol{w}_{\boldsymbol{I}[t]} = \frac{z(t)}{\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1}} \boldsymbol{C}_{\boldsymbol{I}[t]} \boldsymbol{F}[t] \tag{4.12}$$

where we divide by $\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1}$ so as to exactly satisfy the self-financing constraint $\boldsymbol{x}_{\boldsymbol{I}[t]}^\top \mathbf{1} = z(t)$. When δ and p_b in (4.6) are small enough, the effect of this modification on performances will be very small as we will see in Section 5.3.

To implement the optimization problem (4.11), we need to compute $\mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{f}(t)^\top\right)$, $\mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{F}[s]^\top\right)$ and $\mathrm{E}_t^*\left(\boldsymbol{F}[t]\right)$. From the definition of $\boldsymbol{F}[t]$, the problems are reduced to computing $\mathrm{E}_t^*\left(\boldsymbol{f}(r)\right)$ and $\mathrm{E}_t^*\left(\boldsymbol{f}(r)\boldsymbol{f}(u)^\top\right)$ which are explicitly given in the next proposition.

Proposition 4.3. For $t \geq 2$ and $2 \leq r, u \leq t$,

$$\mathrm{E}_t^*\left(\boldsymbol{f}(r)\right) = \boldsymbol{\Psi}_{I(2:r)}\boldsymbol{f}(1) + \sum_{s=2}^{r} \boldsymbol{\Psi}_{I(s+1:r)}\boldsymbol{\mu}_{I(s)} \tag{4.13}$$

and

$$\mathrm{E}_t^*\left(\boldsymbol{f}(r)\boldsymbol{f}(u)^\top\right) = \mathrm{E}_t^*\left(\boldsymbol{f}(r)\right)\mathrm{E}_t^*\left(\boldsymbol{f}(u)\right)^\top + \sum_{a=2}^{\min(r,u)} \boldsymbol{\Psi}_{I(a+1:r)}\boldsymbol{\Sigma}_{I(a)}\boldsymbol{\Psi}_{I(a+1:u)}^\top \tag{4.14}$$

where

$$\boldsymbol{\Psi}_{I(s:t)} = \begin{cases} \boldsymbol{\Phi}_{I(t)} \times \boldsymbol{\Phi}_{I(t-1)} \times \cdots \times \boldsymbol{\Phi}_{I(s)}, & s \leq t \\ \boldsymbol{I}_M, & s = t+1. \end{cases}$$

Proof. See Appendix A.2. □

5. Numerical Experiments

To check investment efficacy of LRS subject to self-financing and short sales constraints under regime switches, we conduct numerical experiments using parameters estimated from market data. Performance comparisons with a myopic optimization and an optimization without considering transaction costs are also provided.

5.1. Model parameters

The model consists of four assets, two fixed incomes and two equities, and two factors that predict asset returns. Two fixed incomes are the US 10 year treasury bond (TSY for short) and the US investment grade corporate bonds (IGC) cited from the Board of Governors of the Federal Reserve System and Goldman Sachs Asset Management. Two equity assets are the Russell 3000 Growth (R3G) and the Russell 3000 Value (R3V) cited from the Bloomberg. The four assets tend to comprise diversified investment portfolios in practice. For factors, we employ a term spread (US 10 year interest rate less US 3 year interest rate, TS) and a default spread (US corporate Baa interest rate less US corporate Aaa interest rate, DS). The four interest rates comprising the two factors are cited from The Board of Governors of the Federal Reserve System. The data set contains 444 monthly data from February 1979 to January 2016. The number of regimes is set to two since, as shown in many empirical researches, a two regime model well describes state changes of the market between bull and bear.

Tables 1 and 2 summarize the estimated parameters in (2.1) and (2.2), respectively. The Akaike's Information Criterion of the two regime model is 5575.3 which is much better than 6641.9 of the single regime model, indicating superior descriptive power of the regime switching model. A notable feature observed in these tables is that variance terms in $\boldsymbol{W}_2$

Table 1: Estimated parameters of $\boldsymbol{L}_i$ and $\boldsymbol{W}_i$ in (2.1) for assets

Regime 1	$\boldsymbol{L}_1$		$\boldsymbol{W}_1$ $(\times 10^{-3})$			
	TS	DS	TSY	IGC	R3G	R3V
TSY	.119	.047	.141	$\underline{.955}$	$\underline{.187}$	$\underline{.204}$
IGC	.243	.055	.153	.182	$\underline{.291}$	$\underline{.332}$
R3G	−.086	1.359	.079	.140	1.275	$\underline{.871}$
R3V	.232	.998	.073	.135	.937	.908
Regime 2	$\boldsymbol{L}_2$		$\boldsymbol{W}_2$ $(\times 10^{-3})$			
	TS	DS	TSY	IGC	R3G	R3V
TSY	.163	.168	.443	$\underline{.786}$	$\underline{-.030}$	$\underline{.015}$
IGC	.172	.251	.490	.878	$\underline{.265}$	$\underline{.320}$
R3G	−.107	−.096	−.043	.547	4.860	$\underline{.835}$
R3V	−.405	.070	.019	.556	3.409	3.434

Diagonal and lower triangular elements of $\boldsymbol{W}_i$ are (co)variances and upper triangular elements with underline denote correlation.

Table 2: Estimated parameters of $\boldsymbol{\mu}_i$, $\boldsymbol{\Phi}_i$ and $\boldsymbol{\Sigma}_i$ in (2.2) for factors

Regime 1	$\boldsymbol{\mu}_1(\times 10^{-3})$	$\boldsymbol{\Phi}_1$		$\boldsymbol{\Sigma}_1$ $(\times 10^{-5})$	
		TS	DS	TS	DS
TS	−.333	.990	.034	.087	$\underline{-.055}$
DS	.336	−.011	.967	−.003	.029
Regime 2	$\boldsymbol{\mu}_2(\times 10^{-3})$	$\boldsymbol{\Phi}_2$		$\boldsymbol{\Sigma}_2$ $(\times 10^{-5})$	
		TS	DS	TS	DS
TS	.506	.959	.004	.614	$\underline{.163}$
DS	1.481	−.015	.918	.082	.411

Diagonal and lower triangular elements of $\boldsymbol{\Sigma}_i$ are (co)variances and upper triangular elements with underline denote correlation.

and $\boldsymbol{\Sigma}_2$ are several times larger than those in $\boldsymbol{W}_1$ and $\boldsymbol{\Sigma}_1$, implying Regime 1 represents rather tranquil state of the market while Regime 2 is a turbulent state.

Transition probabilities between two regimes are estimated as

$$\boldsymbol{P} = \begin{bmatrix} .913 & .087 \\ .177 & .823 \end{bmatrix}. \tag{5.1}$$

On average, Regime 1 continues $1/(1-0.913) = 11.5$ months and Regime 2 continues $1/(1-0.823) = 5.7$ months. Figure 1 shows the time series of the filtered probabilities of Regime 2 which suggests the usefulness of the regime switching model in investment decision making as the model grasps drastic and sudden changes in the market. For example,

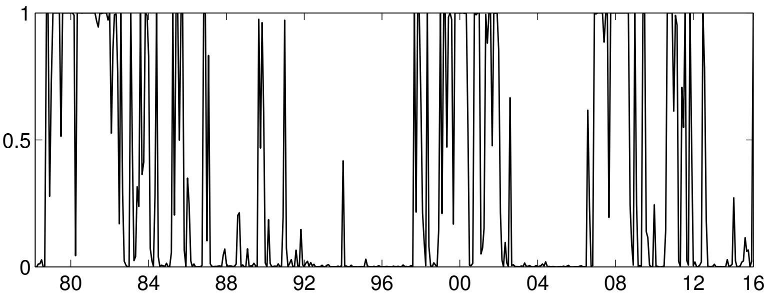

Figure 1: Filtered probabilities of Regime 2

turbulent Regime 2 appears around 2000 when Russian and Asian currency crisis took place followed by the technology bubble and burst. Regime 2 is also tightly related to credit bubble and burst followed by the Lehman Shock around 2008.

It is also worth noting that the estimated regime probabilities are close to either 0 or 1 throughout the estimation period. This makes it possible for an investor to identify the past and current regimes $\boldsymbol{I}[t]$ accurately. Moreover, the regime process shows a strong tendency to stay in the same regime as the self-transition probabilities $p_{1,1} = 0.913$ and $p_{2,2} = 0.823$ in (5.1) are close to 1. These facts suggest a plausible prediction of $I(t+1)$ at t

$$\widehat{I}(t+1) = \mathrm{argmax}_{j\in\mathcal{J}} \left\{ \sum_{i=1}^{J} p(I(t)=i)p_{i,j} \right\}$$

where $p(I(t)=i)$ is the estimated filtered probability. Concerning that both $p(I(t)=i)$ and $p_{i,j}$ are close to either 0 or 1, we expect that $\sum_{i=1}^{J} p(I(t)=i)p_{i,\widehat{I}(t+1)}$ is close to 1. These considerations support the validity of the assumptions in Section 2.

5.2. Evaluation of investment performances

We explain how to measure the investment performance of LRS. A similar procedure will be used for other strategies to compare investment efficacy in Section 5.3.

1. Initial condition:
   An initial portfolio is set to $\boldsymbol{x}(0) = (1,\ldots,1)^\top$. An initial regime is sampled according to the stationary distribution with respect to the transition probability matrix $\boldsymbol{P}$ in (5.1). Similarly, an initial factor $\boldsymbol{f}(1)$ is given as its time stationary mean $\boldsymbol{f}(1) = (\boldsymbol{I}_M - \boldsymbol{\Phi}_{I(1)})^{-1}\boldsymbol{\mu}_{I(1)}$ on the sampled regime $I(1)$.
2. Sampling of $\{I(t)\}$, $\{\boldsymbol{f}(t)\}$ and $\{\boldsymbol{r}(t)\}$:
   Following the initial regime $I(1)$, a sample of the subsequent regime process $\{I(t)\}$ is sampled based upon the transition probability matrix $\boldsymbol{P}$. Given $\{I(t)\}$, a factor process $\{\boldsymbol{f}(t)\}$ is sampled according to (2.2) where $\boldsymbol{\epsilon}_{I(t)}(t)$ is generated from $\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}_{I(t)})$. A return process $\{\boldsymbol{r}(t)\}$ is sampled from (2.1) where $\boldsymbol{u}_{I(t)}(t)$ is generated from $\mathcal{N}(\mathbf{0},\boldsymbol{W}_{I(t)})$. To get samples that are stationary in time, we simulate samples of length 10^4 and use the last 240 steps for performance evaluation whereas the first 9760 steps are abandoned as burn-in. The length of the evaluation period $T_{sim} = 240$ represents a 20 year period in monthly data.

3. Computation of LRS:
At time t, an investor observes the samples of $\boldsymbol{f}[t]$ and $\boldsymbol{I}[t]$, and solves (4.11) to compute the optimal LRS. A plan of investment decisions (3.4) covers from $\boldsymbol{w}_{\boldsymbol{I}[t]}$, a position at t up to $\boldsymbol{w}_{\boldsymbol{I}[t+T-1]}$, a T steps ahead future portfolio. If the short sales constraint $\boldsymbol{w}_{\boldsymbol{I}[t+s]} \geq \mathbf{0}$ is violated for some s, we implement $\boldsymbol{w}_{\boldsymbol{I}[t]}, \ldots, \boldsymbol{w}_{\boldsymbol{I}[t+s-1]}$ whereas the rest are abandoned. Instead, we re-conduct the optimization at $t+s$ to get new $\boldsymbol{w}_{\boldsymbol{I}[t+s]}, \ldots, \boldsymbol{w}_{\boldsymbol{I}[t+s+T-1]}$.
4. Model predictive control:
In a multi-period investment with a finite horizon, obtained strategies are not necessarily fully utilized. It is often the case that, among investment decisions $\boldsymbol{w}_{\boldsymbol{I}[t]}, \ldots, \boldsymbol{w}_{\boldsymbol{I}[t+T-1]}$ computed at t, an investor only uses first τ ($< T$) decisions $\boldsymbol{w}_{\boldsymbol{I}[t]}, \ldots, \boldsymbol{w}_{\boldsymbol{I}[t+\tau-1]}$ and ignores those in the remaining periods. Instead, he or she conducts a next optimization at $t+\tau$ based on updated observations $\boldsymbol{f}[t+\tau]$ and $\boldsymbol{I}[t+\tau]$, and obtains new portfolios $\boldsymbol{w}_{\boldsymbol{I}[t+\tau]}, \ldots, \boldsymbol{w}_{\boldsymbol{I}[t+\tau+T-1]}$ in which only first τ positions are used. This type of optimization is sometimes called a Model Predictive Control (MPC for short). In the following experiments, we vary from $\tau = 1$ to maximum T for comparison.
5. Performance measures:
For measuring investment efficacy, we employ Sharpe ratio as well as utility. The total amount of the portfolio at t penalized for transaction costs is

$$z(t) = \boldsymbol{x}_{\boldsymbol{I}[t-1]}^\top(\mathbf{1} + \boldsymbol{r}(t)) - \frac{1}{2}(\boldsymbol{x}_{\boldsymbol{I}[t-1]} - \boldsymbol{x}_{\boldsymbol{I}[t-2]})^\top \boldsymbol{B}_{I(t)}(\boldsymbol{x}_{\boldsymbol{I}[t-1]} - \boldsymbol{x}_{\boldsymbol{I}[t-2]}) \tag{5.2}$$

with the portfolio $\boldsymbol{x}_{\boldsymbol{I}[t]}$ given in (4.12). The realized net return is then $R(t) = \frac{z(t)}{z(t-1)} - 1$, and net Sharpe ratio ($SR$) and net utility ($U$) are defined by

$$\begin{aligned} SR &= \frac{\mu}{\sigma} \\ U &= \mu - \frac{\lambda}{2}\sigma^2 \end{aligned}$$

where

$$\begin{aligned} \mu &= \frac{1}{T_{sim}} \sum_{t=1}^{T_{sim}} R(t) \\ \sigma^2 &= \frac{1}{T_{sim} - 1} \sum_{t=1}^{T_{sim}} (R(t) - \mu)^2. \end{aligned}$$

6. We repeat Steps 1∼5 above N_{sim}=100 times. To improve accuracy of samples performance measures, we apply 2 types of antithetic variates $\boldsymbol{u}_{I(t)}(t)$ and $-\boldsymbol{u}_{I(t)}(t)$ when sampling $\boldsymbol{r}(t)$ in Step 2. This means that we obtain $100 \times 2 = 200$ samples of SR and U and then compute sample averages and confidence intervals.

The optimization algorithm in Step 3 is implemented by CVX Version 2.1 developed by Grant and Boyd [10] which works on MATLAB R2016b and efficiently solves the quadratic programming subject to the second order cone constraints.

5.3. Comparison of investment performances

In the experiments of LRS, we use the following parameters. The discount rate is fixed to $\rho = 1$ and the risk aversion coefficient is $\lambda = 1$. The parameters of the self-financing constraint are $\delta = 0.025$ and $p_b = 0.05$, and that of the short sales constraint is set to $p_s = 0.05$. The MPC is examined by changing the number of steps τ to utilize the optimization decisions from 1 up to T. As mentioned in Section 5.2, the optimization under MPC is conducted in every τ steps. The transaction cost matrix for regime i is set to $\boldsymbol{B}_i = \frac{1}{5}\mathrm{Diag}(w_i^1, \ldots, w_i^N)$, a diagonal matrix where w_i^n is the square root of an n-th diagonal of $\boldsymbol{W}_i$. Since $\boldsymbol{W}_2$ is a couple times larger than $\boldsymbol{W}_1$ as shown in Table 1, we assume the higher transaction costs for the higher risk assets for investors to pay.

Table 3 exhibits sample averages and 95% confidence intervals of net Sharpe ratio SR and net utility U for $T = 1, 3, 5$. Looking out in a horizontal direction, myopic solutions at

Table 3: Investment performances of LRS

	SR		
$\tau \setminus T$	1	3	5
1	.2349	.2431	.2448
	.2254/.2444	.2338/.2525	.2355/.2541
3		.2390	.2432
		.2297/.2482	.2340/.2525
5			.2410
			.2318/.2503

	U $(\times 10^{-2})$		
$\tau \setminus T$	1	3	5
1	.8210	.8318	.8332
	.7842/.8578	.7958/.8679	.7973/.8690
3		.8258	.8302
		.7895/.8621	.7943/.8660
5			.8285
			.7924/.8646

Sample average at upper levels and 95% confidence interval at lower levels. $\lambda = 1, \rho = 1, \delta = 0.025, p_b = p_s = 0.05, \boldsymbol{B}_i = \frac{1}{5}\mathrm{Diag}(w_i^1, \ldots, w_i^N)$ where w_i^n is the square root of an n-th diagonal of $\boldsymbol{W}_i$.

$T = 1$ perform the most poorly across the ranges of the investment horizon. The longer the horizon, the better investment efficacy is in both SR and U. This is a natural consequence since, for large T, investment decisions are made based on long term fluctuations of the return process including future possibility of regime switches. On the other hand, looking out impacts of τ, $\tau = 1$ exhibits the best investment performances in terms of SR and U across a range of T. Since the optimization is conducted every τ steps in MPC, newly observed data are used in the optimization more frequently for smaller τ. The results means that the latest data bring more valuable information from a perspective of SR and U to the multi-period optimization than taking over previously optimized solutions in longer time intervals. We remark that $\tau = 1$ reflects information predicted toward T time steps ahead and thus should be distinguished from a myopic decision for $T = 1$.

To check robustness of the results, Table 4 shows sample averages and 95% confidence intervals of SR when we change one of the parameters to either $\lambda = 10$ or $\delta = 0.05$. Other parameters are the same as in Table 3. We observe that SR shows better performances for larger T and smaller τ as in Table 3. Especially, SR at the lower panel for $\delta = 0.05$ are very close to those in Table 3. Since δ in (4.6) controls how strictly the stochastic self-financing

Table 4: SR for $\lambda = 10$ and $\delta = 0.05$

$\tau \setminus T$	$\lambda = 10$		
	1	3	5
1	.2726	.2764	.2793
	.2631/.2821	.2669/.2859	.2698/.2889
3		.2746	.2778
		.2651/.2841	.2682/.2873
5			.2765
			.2670/.2859

$\tau \setminus T$	$\delta = 0.05$		
	1	3	5
1	.2349	.2431	.2448
	.2254/.2444	.2338/.2525	.2355/.2541
3		.2387	.2430
		.2295/.2480	.2337/.2522
5			.2412
			.2319/.2504

Sample average at upper levels and 95% confidence interval at lower levels. $\rho = 1, p_b = p_s = 0.05, \boldsymbol{B}_i = \frac{1}{5}\text{Diag}(w_i^1, \ldots, w_i^N)$ where w_i^n is the square root of an n-th diagonal of $\boldsymbol{W}_i$ at both panels. $\delta = 0.025$ at upper panel and $\lambda = 1$ at lower panel.

constraint is satisfied, this indicates that investment performances are robust even when the self-financing constraint is relaxed.

As both SR and U in Table 3 continue to improve as T increases up to 5, this motivates us to explore to see if the investment efficacy still continues to improve for T greater than 5. The number of variables in the optimization given in (3.5) increases exponentially fast as T increases, which makes our experiments time wasting even for moderate values of T. On the other hand, the probability $p(\boldsymbol{I}[t])$ becomes very small for most sample paths of the regime process $\boldsymbol{I}[t]$ since transition probabilities from one regime to another are significantly smaller than that of a self-transition. This implies that the optimization problem would not be affected very much even when we ignore regime transitions with small probabilities. For moderate to large values of T, we therefore restrict the sample space of $\boldsymbol{I}[t]$ to those with at most K regime switches in the time interval $[1, T]$. The maximum number of sample paths to deploy is J^{t-1} for $K = T - 1$ which is reduced to $t(t-1)/2 + 1$ for $K = 2$ and t for $K = 1$, respectively. The upper panel of Table 5 shows the number of variables in the optimization. Compared with unrestricted case for $K = T - 1$, the number of variables is drastically reduced especially for $K = 1$. The lower panel of Table 4 summarizes the probability that the number of regime switches is at most K. Even for $T = 9$, about 91.4% (74.6%, respectively) of sample paths contain at most $K = 2$ ($K = 1$) regime switches, as we expected.

Tables 6 summarizes the investment performances for T up to 9 and $K = 1, 2, T - 1$. We fix to $\tau = 1$ as it achieves the best results and other parameters are the same as in Table 3. We observe that the performances are not very different even when the sample space is restricted in the optimization. To see more in detail, Figure 2 compares cumulative

Table 5: Number of variables and probability coverages under sample space restriction

$K \setminus T$	Number of variables 1	3	5	7	9
$T-1$	4	108	908	5,644	30,732
2	4	108	700	2,548	6,804
1	4	88	380	1,008	2,100

$K \setminus T$	Probability coverages 1	3	5	7	9
$T-1$	1	1	1	1	1
2	1	1	.990	.962	.914
1	1	.985	.923	.838	.746

Table 6: Investment performances under sample space restriction

$K \setminus T$	*SR* 1	3	5	7	9
$T-1$	.2349	.2431	.2448		
	.2254/.2444	.2338/.2525	.2355/.2541		
2	.2349	.2431	.2453	.2462	
	.2254/.2444	.2338/.2525	.2360/.2546	.2369/.2555	
1	.2349	.2441	.2476	.2495	.2507
	.2254/.2444	.2348/.2535	.2383/.2569	.2402/.2588	.2414/.2600

$K \setminus T$	$U \quad (\times 10^{-1})$ 1	3	5	7	9
$T-1$	.8210	.8318	.8332		
	.7842/.8578	.7958/.8679	.7973/.8690		
2	.8210	.8318	.8319	.8317	
	.7842/.8578	.7958/.8679	.7962/.8676	.7962/.8672	
1	.8210	.8300	.8269	.8248	.8235
	.7842/.8578	.7942/.8658	.7919/.8619	.7903/.8593	.7894/.8576

Sample average at upper levels and 95% confidence interval at lower levels. $\tau = 1, \lambda = 1, \rho = 1, \delta = 0.025, p_b = p_s = 0.05, \boldsymbol{B}_i = \frac{1}{5}\text{Diag}(w_i^1, \ldots, w_i^N)$ where w_i^n is the square root of an n-th diagonal of $\boldsymbol{W}_i$.

weights of LRS for $K = 1$ (left) and $K = 4$ (right) when $T = 5$. Three lines from bottom

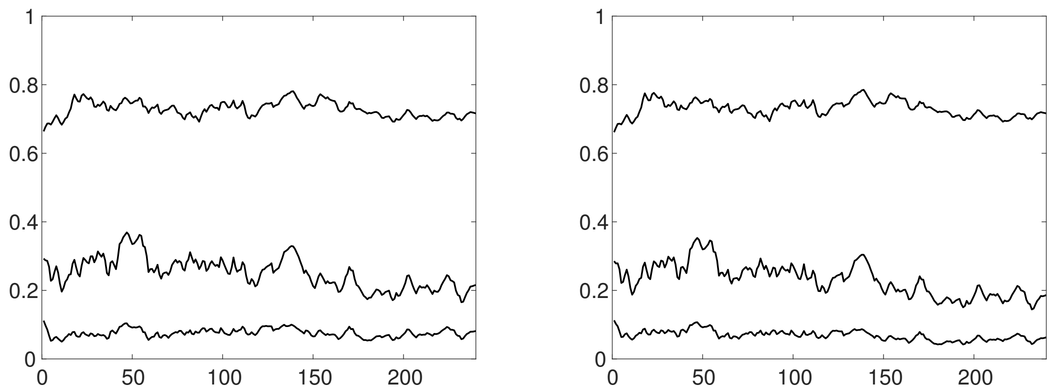

Figure 2: Cumulative weights of LRS for $K = 1$ (left) and $K = T - 1 = 4$ (right)

to top respectively show sample averages of $w_{\boldsymbol{I}[t],1}$, $w_{\boldsymbol{I}[t],1} + w_{\boldsymbol{I}[t],2}$, $w_{\boldsymbol{I}[t],1} + w_{\boldsymbol{I}[t],2} + w_{\boldsymbol{I}[t],3}$ for $t = 1 \sim 240$. Since portfolio weights for $K = 1$ are close to those for $K = 4$, this explains similar performances in Table 6. We also observe in Table 6 that SR continues to improve for T beyond 3 for $K = 1, 2$ and are nearly saturated around $T = 9$. In case of $\tau = 1$, an investor only uses $\boldsymbol{w}_{\boldsymbol{I}[t]}$ among all solutions $\boldsymbol{w}_{\boldsymbol{I}[t]}, \ldots, \boldsymbol{w}_{\boldsymbol{I}[t+T-1]}$ obtained at t, and conducts a new optimization at $t + 1$. The saturation of the performance measures is therefore a natural consequence of that $\boldsymbol{w}_{\boldsymbol{I}[t]}$ is not different very much for large T. In sum, the results indicate the effectiveness of the variable reduction method as it makes the optimization problem numerically tractable for larger T to acquire better investment performances.

Finally, we compare investment performances of LRS with those of the optimal strategy without concerning transaction costs. When an objective function does not contain transaction costs, a multi-period dynamic optimization problem is in principle reduced to a single-period optimization. In our notation, this single-period optimization problem without transaction costs (w/o TC for short) is formulated by letting $T = 1$ and $U_3(t) = 0$ in (2.4). The self-financing and the short sales constraints are applied to w/o TC, too. Table 7 summarizes sample averages and 95% confidence intervals of investment performances of both LRS and w/o TC. Both SR and U of w/o TC are calculated by the same way as LRS, i.e., transaction costs are deducted ex post in (5.2). Due to the lack of information over multi-period horizon in the future including transaction costs, w/o TC significantly underperforms the LRS.

6. Concluding Remarks

In this paper, we extend the linear rebalancing rule proposed in Moallemi and Sağlam [17] to regime switching models and provide a dynamic investment strategy maximizing a mean-variance utility under practical conditions. We also propose a sample space reduction method to suppress rapid increase of a number of optimization variables as an investment horizon extends. Numerical experiments for realistic circumstances subject to the short sales constraints and the self-financing constraints show that the reduction method works efficiently and the proposed strategy achieves satisfactory investment performances.

As mentioned, a difficulty arising in multi-period portfolio optimization is a state space explosion. When an investor attempts to obtain a dynamic strategy, the problem becomes more complicated since future investment decisions should be made according to up-to-date

Table 7: Performance comparisons of LRS and w/o TC

	SR	U ($\times 10^{-2}$)
w/o TC	.1797	.0730
	.1689/.1906	.0680/.0780
LRS	.2349	.8210
	.2254/.2444	.7842/.8578

Sample average at upper levels and 95% confidence interval at lower levels. $T = \tau = 1, \lambda = 1, \rho = 1, \delta = 0.025, p_b = p_s = 0.05, \boldsymbol{B}_i = \frac{1}{5}\mathrm{Diag}(w_i^1, \ldots, w_i^N)$ where w_i^n is the square root of an n-th diagonal of $\boldsymbol{W}_i$.

observations. The idea of LRS is to overcome these difficulties by restricting the strategy space to the set of strategies represented by a linear combination of factors up to the time of investment. Although LRS is not exactly optimal in all possible dynamic strategies, judging not only from our experiments where we confirm that LRS provides solutions close to optimal but from its theoretical basis that LRS is optimal for an infinite horizon problem without constraints as solved in Gârleanu and Pedersen [9] and Komatsu and Makimoto [12], we conclude that LRS is potentially applicable to a wide class of complex dynamic portfolio optimizations under practical investment conditions.

Future research is planned on a couple of front of more advanced models. For example, extensions of LRS to include more complex factor dynamics and constraints imposing targeted levels of the expected returns and/or the risk penalty.

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A. Appendix

A.1. Computational procedure of $\xi_{\boldsymbol{I}[t]}$

We approximate the amount of the portfolio $z(t)$ by the expected portfolio value $\xi_{\boldsymbol{I}[t]}$ obtained by the following iterative procedure.

1. For initial portfolio $\boldsymbol{x}(0)$ and observed $\boldsymbol{r}(1)$, we set the initial values:

$$\xi_{\boldsymbol{I}[0]} = \boldsymbol{x}(0)^\top \mathbf{1}, \qquad \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[0]} = \frac{\boldsymbol{x}(0)}{\xi_{\boldsymbol{I}[0]}}, \qquad \xi_{\boldsymbol{I}[1]} = \boldsymbol{x}(0)^\top \{\mathbf{1} + \boldsymbol{r}(1)\}. \tag{A.1}$$

2. For $t = 1, \ldots, T-1$:
   (a) For each $\boldsymbol{I}[t]$, we predict $I(t+1)$ and solve the following optimization problem to get the optimal solution $\widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t]}$.

$$\begin{aligned} \max_{\widetilde{\boldsymbol{w}}} \quad & \widetilde{\boldsymbol{w}}^\top \boldsymbol{L}_{I(t+1)} \mathrm{E}_t^* \left(\boldsymbol{f}(t)\right) - \frac{\lambda}{2} \widetilde{\boldsymbol{w}}^\top \boldsymbol{W}_{I(t+1)} \widetilde{\boldsymbol{w}} \\ & - \frac{\xi_{\boldsymbol{I}[t]}}{2} \left\{ \widetilde{\boldsymbol{w}} - \frac{\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}} \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t-1]} \right\}^\top \boldsymbol{B}_{I(t+1)} \left\{ \widetilde{\boldsymbol{w}} - \frac{\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}} \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t-1]} \right\} \\ \text{s. t.} \quad & \widetilde{\boldsymbol{w}}^\top \mathbf{1} = 1, \quad \widetilde{\boldsymbol{w}} \geq 0 \end{aligned}$$

   (b) For each $\boldsymbol{I}[t+1] = \{\boldsymbol{I}[t], I(t+1)\}$, we compute

$$\begin{aligned} \xi_{\boldsymbol{I}[t+1]} \;=\; & \xi_{\boldsymbol{I}[t]} \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t]} \{\mathbf{1} + \boldsymbol{L}_{I(t+1)} \mathrm{E}_t^* \left(\boldsymbol{f}(t)\right)\} \\ & - \frac{\xi_{\boldsymbol{I}[t]}^2}{2} \left\{ \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t]} - \frac{\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}} \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t-1]} \right\}^\top \boldsymbol{B}_{I(t+1)} \left\{ \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t]} - \frac{\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}} \widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t-1]} \right\}. \end{aligned}$$

Step 2(a) solves the optimal weights $\widetilde{\boldsymbol{w}}_{\boldsymbol{I}[t]}$ for a single-period mean-variance optimization subject to transaction costs and self-financing/no short sales constraints. Thus, $\xi_{\boldsymbol{I}[t+1]}$ in Step 2(b) denotes the expected value of the portfolio for $\boldsymbol{I}[t+1]$ subject to the transaction cost. Since both LRS and the above procedure determine the portfolio weights from $\boldsymbol{I}[t]$, we expect that $\xi_{\boldsymbol{I}[t]}$ gives a reasonable approximation to the portfolio value $z(t)$ under LRS. We also note that, since Step 2(a) is just a single-period optimization, computational burden of this procedure is low.

A.2. Proofs

We will prove Propositions 4.1 to 4.3 in Section 4. In what follows, $\mathcal{F}(t)$ denotes the filtration generated by $\boldsymbol{I}[t]$. We first show a preliminary lemma.

Lemma A.1. Let $\boldsymbol{A}$ be an $\mathcal{F}(t)$-measurable matrix of size $N \times (1 + (t-1)M)$ and define $\boldsymbol{y}[t] = \boldsymbol{A}\boldsymbol{F}[t]$. Then, for any $\mathcal{F}(t)$-measurable matrix $\boldsymbol{Q}$ of size $N \times N$, we obtain

$$\mathrm{E}_t^* \left(\boldsymbol{y}[t]^\top \boldsymbol{Q} \boldsymbol{y}[t]\right) = \mathrm{vec}(\boldsymbol{A}^\top)^\top \left\{\boldsymbol{Q} \otimes \mathrm{E}_t^* \left(\boldsymbol{F}[t]\boldsymbol{F}[t]^\top\right)\right\} \mathrm{vec}(\boldsymbol{A}^\top). \tag{A.2}$$

Proof. Let $\boldsymbol{a}_n^\top$ denote an n-th row vector of $\boldsymbol{A}$ and let $q_{n,m}$ denote (n,m)-element of $\boldsymbol{Q}$. Then

$$\begin{aligned} \mathrm{E}_t^* \left(\boldsymbol{y}[t]^\top \boldsymbol{Q} \boldsymbol{y}[t]\right) &= \mathrm{E}_t^* \left(\boldsymbol{F}[t]^\top \boldsymbol{A}^\top \boldsymbol{Q} \boldsymbol{A} \boldsymbol{F}[t]\right) \\ &= \sum_{n=1}^N \sum_{m=1}^N \mathrm{E}_t^* \left(\boldsymbol{F}[t]^\top \boldsymbol{a}_n q_{n,m} \boldsymbol{a}_m^\top \boldsymbol{F}[t]\right) \\ &= \sum_{n=1}^N \sum_{m=1}^N q_{n,m} \boldsymbol{a}_n^\top \mathrm{E}_t^* \left(\boldsymbol{F}[t]\boldsymbol{F}[t]^\top\right) \boldsymbol{a}_m \end{aligned}$$

which can be rewritten as (A.2). □

Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. We first note that, since $I(t+1)$ is predictable at t under the assumption in Section 2, variables determined by $I(t+1)$ such as $\boldsymbol{L}_{I(t+1)}$ are also $\mathcal{F}(t)$-measurable. Then, $\mathrm{E}\left(\boldsymbol{w}_{I(1)}^\top \boldsymbol{L}_{I(2)}\boldsymbol{f}(1)\middle|\, \boldsymbol{x}(0), H(1)\right) = \boldsymbol{C}_{\boldsymbol{I}[1]}^\top \boldsymbol{L}_{I(2)}\boldsymbol{f}(1)$ holds for $t=1$. Substituting $\boldsymbol{w}_{\boldsymbol{I}[t]} = \boldsymbol{C}_{\boldsymbol{I}[t]}\boldsymbol{F}[t]$ and conditioning on $\boldsymbol{I}[t]$, we obtain

$$\begin{aligned}
\mathrm{E}\left(\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t)\middle|\, \boldsymbol{x}(0), H(1)\right) &= \mathrm{E}\left(\boldsymbol{F}[t]^\top \boldsymbol{C}_{\boldsymbol{I}[t]}^\top \boldsymbol{L}_{I(t+1)}\boldsymbol{f}(t)\middle|\, \boldsymbol{x}(0), H(1)\right) \\
&= \mathrm{E}\left(\boldsymbol{F}[t]^\top \sum_{n=1}^{N} \boldsymbol{c}_{\boldsymbol{I}[t],n}\boldsymbol{\ell}_{I(t+1),n}^\top \boldsymbol{f}(t)\middle|\, \boldsymbol{x}(0), H(1)\right) \\
&= \sum_{\boldsymbol{I}[t]} p(\boldsymbol{I}[t])\mathrm{E}_t^*\left(\sum_{n=1}^{N} \boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \boldsymbol{F}[t]\boldsymbol{f}(t)^\top \boldsymbol{\ell}_{I(t+1),n}\right) \\
&= \sum_{\boldsymbol{I}[t]} p(\boldsymbol{I}[t])\sum_{n=1}^{N} \boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{f}(t)^\top\right)\boldsymbol{\ell}_{I(t+1),n} \\
&= \sum_{\boldsymbol{I}[t]} p(\boldsymbol{I}[t])\mathrm{vec}(\boldsymbol{C}_{\boldsymbol{I}[t]}^\top)^\top \{\boldsymbol{I}_N \otimes \mathrm{E}_t^*\left(\boldsymbol{F}[t]\boldsymbol{f}(t)^\top\right)\}\mathrm{vec}(\boldsymbol{L}_{I(t+1)}^\top)
\end{aligned}$$

where $\boldsymbol{c}_{\boldsymbol{I}[t],n}^\top$ and $\boldsymbol{\ell}_{I(t+1),n}^\top$ respectively denote an n-th row vector of $\boldsymbol{C}_{\boldsymbol{I}[t]}$ and $\boldsymbol{L}_{I(t+1)}$. This proves (4.1). (4.2) is derived by conditioning on $\boldsymbol{I}[t]$ and plugging $\boldsymbol{A} = \boldsymbol{C}_{\boldsymbol{I}[t]}$ and $\boldsymbol{Q} = \boldsymbol{W}_{I(t+1)}$ into (A.2). By approximating $z(t)$ by $\xi_{\boldsymbol{I}[t]}$, (2.7) becomes

$$U_3(t) = \mathrm{E}\left(\xi_{\boldsymbol{I}[t]}\Delta\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \boldsymbol{B}_{I(t+1)}\Delta\boldsymbol{w}_{\boldsymbol{I}[t]}\middle|\, \boldsymbol{x}(0), H[1]\right), \qquad \Delta\boldsymbol{w}_{\boldsymbol{I}[t]} = \boldsymbol{w}_{\boldsymbol{I}[t]} - \frac{\xi_{\boldsymbol{I}[t-1]}}{\xi_{\boldsymbol{I}[t]}}\boldsymbol{w}_{\boldsymbol{I}[t-1]}.$$

Since $\Delta\boldsymbol{w}_{\boldsymbol{I}[t]} = \Delta\boldsymbol{C}_{\boldsymbol{I}[t]}\boldsymbol{F}[t]$, (4.3) is derived by conditioning on $\boldsymbol{I}[t]$ and substituting $\boldsymbol{A} = \Delta\boldsymbol{C}_{\boldsymbol{I}[t]}$ and $\boldsymbol{Q} = \boldsymbol{B}_{I(t+1)}$ into (A.2). □

Proof of Proposition 4.2. By recursively solving (2.2), we obtain

$$\boldsymbol{f}(r) = \boldsymbol{\Psi}_{I(2:r)}\boldsymbol{f}(1) + \sum_{s=2}^{r} \boldsymbol{\Psi}_{I(s+1:r)}\boldsymbol{\mu}_{I(s)} + \sum_{s=2}^{r} \boldsymbol{\Psi}_{I(s+1:r)}\boldsymbol{\epsilon}_{I(s)}(s). \tag{A.3}$$

Conditioned on $\boldsymbol{f}(1)$ and $\boldsymbol{I}[t]$, $\boldsymbol{f}(t)$ follows a multivariate normal distribution and so is $\boldsymbol{w}_{\boldsymbol{I}[t]} = \boldsymbol{C}_{\boldsymbol{I}[t]}\boldsymbol{F}[t]$. The conditional mean and the covariance are calculated as

$$\mathrm{E}_t^*\left(\boldsymbol{w}_{\boldsymbol{I}[t]}\right) = \boldsymbol{C}_{\boldsymbol{I}[t]}\mathrm{E}_t^*\left(\boldsymbol{F}[t]\right)$$

and

$$\mathrm{V}_t^*\left(\boldsymbol{w}_{\boldsymbol{I}[t]}\right) = \boldsymbol{C}_{\boldsymbol{I}[t]}\mathrm{E}_t^*\left(\{\boldsymbol{F}[t] - \mathrm{E}_t^*\left(\boldsymbol{F}[t]\right)\}\{\boldsymbol{F}[t] - \mathrm{E}_t^*\left(\boldsymbol{F}[t]\right)\}^\top\right)\boldsymbol{C}_{\boldsymbol{I}[t]}^\top = \boldsymbol{C}_{\boldsymbol{I}[t]}\boldsymbol{\Lambda}_{\boldsymbol{I}[t]}\boldsymbol{C}_{\boldsymbol{I}[t]}^\top$$

where

$$\boldsymbol{\Lambda}_{\boldsymbol{I}[t]} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{I}[t]} & \\ 0 & & & \end{bmatrix}, \quad \widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{I}[t]} = \mathrm{E}_t^*\begin{pmatrix} \boldsymbol{\beta}(2)\boldsymbol{\beta}(2)^\top & \cdots & \boldsymbol{\beta}(2)\boldsymbol{\beta}(t)^\top \\ \vdots & \ddots & \vdots \\ \boldsymbol{\beta}(t)\boldsymbol{\beta}(2)^\top & \cdots & \boldsymbol{\beta}(t)\boldsymbol{\beta}(t)^\top \end{pmatrix}$$

and

$$\boldsymbol{\beta}(r) = \boldsymbol{f}(r) - \mathrm{E}_t^* \left(\boldsymbol{f}(r) \right) = \sum_{a=2}^{r} \boldsymbol{\Psi}_{I(a+1:r)} \boldsymbol{\epsilon}_{I(a)}(a), \qquad r = 2, \ldots, t. \tag{A.4}$$

Note that $\widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{I}[t]}$ is explicitly given from (A.7) below. Let $\widetilde{\boldsymbol{\Theta}}_{\boldsymbol{I}[t]}^\top \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{I}[t]} = \widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{I}[t]}$ be the Cholesky decomposition of $\widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{I}[t]}$ that is positive definite. Then $\boldsymbol{\Theta}_{\boldsymbol{I}[t]}^\top \boldsymbol{\Theta}_{\boldsymbol{I}[t]} = \boldsymbol{\Lambda}_{\boldsymbol{I}[t]}$ where

$$\boldsymbol{\Theta}_{\boldsymbol{I}[t]} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widetilde{\boldsymbol{\Theta}}_{\boldsymbol{I}[t]}^\top & \\ 0 & & & \end{bmatrix}. \tag{A.5}$$

Let $\mu_{\boldsymbol{I}[t]} = \mathrm{E}_t^* \left(\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1} \right)$ and $\sigma_{\boldsymbol{I}[t]}^2 = \mathrm{V}_t^* \left(\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1} \right)$. Then the self-financing constraint (4.6) is represented by $\mu_{\boldsymbol{I}[t]} = 1$ and

$$\mathrm{P}\left(|\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1} - 1| > \delta \right) = \mathrm{P}\left(\left| \frac{\boldsymbol{w}_{\boldsymbol{I}[t]}^\top \mathbf{1} - 1}{\sigma_{\boldsymbol{I}[t]}} \right| > \frac{\delta}{\sigma_{\boldsymbol{I}[t]}} \right) = 2\{1 - \Phi(\delta / \sigma_{\boldsymbol{I}[t]})\} \leq p_b$$

that give (4.8) and (4.9). For the short sales constraint, we note

$$\begin{aligned} \mathrm{E}_t^* \left(w_{\boldsymbol{I}[t],n} \right) &= \boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \mathrm{E}_t^* \left(\boldsymbol{F}[t] \right) \\ \mathrm{V}_t^* \left(w_{\boldsymbol{I}[t],n} \right) &= \boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \boldsymbol{\Lambda}_{\boldsymbol{I}[t]} \boldsymbol{c}_{\boldsymbol{I}[t],n} = \boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \boldsymbol{\Theta}_{\boldsymbol{I}[t]}^\top \boldsymbol{\Theta}_{\boldsymbol{I}[t]} \boldsymbol{c}_{\boldsymbol{I}[t],n}. \end{aligned}$$

For a normal random variable X with mean μ and variance σ^2, $\mathrm{P}\left(X < 0 \right) \leq p$ is equivalent to $\mu \geq \sigma \Phi^{-1}(1-p)$ for $p \in (0, 0.5)$. Substituting $\mu = \boldsymbol{c}_{\boldsymbol{I}[t],n}^\top \mathrm{E}_t^* \left(\boldsymbol{F}[t] \right)$ and $\sigma = \| \boldsymbol{\Theta}_{\boldsymbol{I}[t]} \boldsymbol{c}_{\boldsymbol{I}[t],n} \|_2$ then yields (4.10). □

Proof of Proposition 4.3. Since

$$\mathrm{E}_t^* \left(\boldsymbol{\Psi}_{I(s+1:r)} \boldsymbol{\epsilon}_{I(s)}(s) \right) = \boldsymbol{\Psi}_{I(s+1:r)} \mathrm{E}_t^* \left(\boldsymbol{\epsilon}_{I(s)}(s) \right) = \mathbf{0}, \qquad s \leq r \leq t$$

by the assumption, (A.3) gives (4.13). From (A.4) and $\mathrm{E}_t^* \left(\boldsymbol{\beta}(r) \right) = \mathbf{0}$, we obtain

$$\mathrm{E}_t^* \left(\boldsymbol{f}(r) \boldsymbol{f}(u)^\top \right) = \mathrm{E}_t^* \left(\boldsymbol{f}(r) \right) \mathrm{E}_t^* \left(\boldsymbol{f}(u) \right)^\top + \mathrm{E}_t^* \left(\boldsymbol{\beta}(r) \boldsymbol{\beta}(u)^\top \right). \tag{A.6}$$

Noting that

$$\mathrm{E}_t^* \left(\boldsymbol{\epsilon}_{I(a)}(a) \boldsymbol{\epsilon}_{I(b)}(b)^\top \right) = \begin{cases} \boldsymbol{\Sigma}_{I(a)}, & a = b, \\ \boldsymbol{O}, & a \neq b, \end{cases}$$

the second term in (A.6) is calculated as

$$\begin{aligned} \mathrm{E}_t^* \left(\boldsymbol{\beta}(r) \boldsymbol{\beta}(u)^\top \right) &= \mathrm{E}_t^* \left(\left\{ \sum_{a=2}^{r} \boldsymbol{\Psi}_{I(a+1:r)} \boldsymbol{\epsilon}_{I(a)}(a) \right\} \left\{ \sum_{b=2}^{u} \boldsymbol{\Psi}_{I(b+1:u)} \boldsymbol{\epsilon}_{I(b)}(b) \right\}^\top \right) \\ &= \sum_{a=2}^{\min(r,u)} \boldsymbol{\Psi}_{I(a+1:r)} \mathrm{E}_t^* \left(\boldsymbol{\epsilon}_{I(a)}(a) \boldsymbol{\epsilon}_{I(a)}(a)^\top \right) \boldsymbol{\Psi}_{I(a+1:u)}^\top \\ &= \sum_{a=2}^{\min(r,u)} \boldsymbol{\Psi}_{I(a+1:r)} \boldsymbol{\Sigma}_{I(a)} \boldsymbol{\Psi}_{I(a+1:u)}^\top. \end{aligned} \tag{A.7}$$

This proves (4.14). □

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