# Discrete Decreasingly Minimal Flows

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## 1. Introduction

N. Megiddo [4], [5] introduced and solved the problem of finding a (possibly fractional) maximum flow that is "lexicographically optimal" on the set of edges leaving the source node. This problem was studied further by S. Fujishige [3] as the lexicographically optimal base of a submodular system. In [1, 2], we investigated the discrete counterpart of these problems in the framework of discrete decreasing minimization on an M-convex set. In this paper, we consider another generalization of Megiddo's problem where the lexico-optimality of an integer flow is defined with reference to the flow values on an arbitrary edge subset.

## 2. Dec-min flow problem

Let $D = (V, A)$ be a digraph endowed with integer-valued functions $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ with $f \leq g$. We are given a finite integer-valued function $m$ on $V$ with $\widetilde{m}(V) = 0$, where $\widetilde{m}(X) := \sum[m(v) : v \in X]$ for $X \subseteq V$. An *m-flow* $x$ will mean a finite-valued function on $A$ satisfying

$$\sum[x(uv) : uv \in A] - \sum[x(vu) : vu \in A] = m(v)$$

at each node $v \in V$. An $m$-flow $x$ is called *$(f, g)$-bounded* (or *feasible*) if $f \leq x \leq g$. Let $\ddot{Q} = \ddot{Q}(f, g; m)$ denote the set of $(f, g)$-bounded integral $m$-flows. We assume that $\ddot{Q} \neq \emptyset$.

Let $F$ be a specified subset of $A$. We say that $z \in \ddot{Q}$ is *decreasingly minimal on F* (or *F-dec-min* for short) if the largest component of $z$ in $F$ is as small as possible, within this, the next largest (but not necessarily distinct) component of $z$ in $F$ is as small as possible, and so on.

**Remark 1.** One may also be interested in an integral $(f, g)$-bounded $m$-flow $z$ that is increasingly maximal (inc-max) on $F$ in the sense that the smallest $z$-value on $F$ is as large as possible, within this, the second smallest $z$-value on $F$ is as large as possible, and so on. But an $(f, g)$-bounded $m$-flow $z$ is increasingly maximal on $F$ precisely if $-z$ is a $(-g, -f)$-bounded $(-m)$-flow that is dec-min on $F$. Thus the inc-max and dec-min problems are equivalent. ■

## 3. Existence of a dec-min flow

When there are edges with $f(e) = -\infty$ or $g(e) = +\infty$, it may occur that no dec-min feasible $m$-flow exists. For example, if $D$ is a di-circuit, $F = A$, $m \equiv 0$, $f \equiv -\infty$, and $g \equiv 0$, then $z \equiv k$ is a feasible $m$-flow for each integer $k \leq 0$, and hence there is no $F$-dec-min feasible $m$-flow.

Theorem 1 below gives a characterization for the existence of an $F$-dec-min feasible $m$-flow. Define a digraph $D^{\infty} = (V, A^{\infty})$ by

$$A^{\infty} := \{uv : uv \in A,\ f(uv) = -\infty\} \\ \cup \{vu : uv \in A - F,\ g(uv) = +\infty\}.$$

**Theorem 1.** There exists an $F$-dec-min $(f, g)$-bounded integral $m$-flow if and only if there is no di-circuit $C$ with $C \cap F \neq \emptyset$ in $D^{\infty}$. ■

## 4. Criteria for dec-minimality

We give two characterizations for $z$ to be decreasingly minimal on $F$. One of them refers to the non-existence of an improving di-circuit in an auxiliary digraph, and the other refers to the existence of potential-vectors on the nodes.

Given two $k$-dimensional vectors $\underline{c} = (c_1, c_2, \ldots, c_k)$ and $\underline{d} = (d_1, d_2, \ldots, d_k)$, we say that $\underline{c}$ is *lexicographically smaller* than $\underline{d}$, in notation $\underline{c} \prec \underline{d}$, if $\underline{c} \neq \underline{d}$ and $c_i < d_i$ where $i$ is the first component in which they differ. We write $\underline{c} \preceq \underline{d}$ if $\underline{c} = \underline{d}$ or $\underline{c} \prec \underline{d}$.

Given a digraph $D_0 = (V, A_0)$, let $\underline{c} : A_0 \to \mathbf{R}^k$ be a vector-valued function on the edge-set $A_0$ that assigns a vector $\underline{c}(e) = (c_1(e), c_2(e), \ldots, c_k(e))$ to each edge $e$ of $D_0$. We call a vector-valued function $\underline{\pi} : V \to \mathbf{R}^k$ on $V$ *$\underline{c}$-feasible* if $\underline{\pi}(v) - \underline{\pi}(u) \preceq \underline{c}(uv)$ holds for every edge $uv \in A_0$.

A di-circuit $C$ is said to be $\underline{c}$*-negative* if the sum $\widetilde{\underline{c}}(C) = (\widetilde{c}_1(C), \widetilde{c}_2(C), \dots, \widetilde{c}_k(C))$ of the $\underline{c}$-vectors over $C$ is lexicographically smaller than the $k$-dimensional zero vector $\underline{0}$. We say that $\underline{c}$ is *conservative* if $D_0$ has no $\underline{c}$-negative di-circuit.

The following is a Gallai-type theorem for the lexico-ordering using potential-vectors.

**Theorem 2.** Given a digraph $D_0 = (V, A_0)$ and a vector-valued function $\underline{c} : A_0 \to \mathbf{R}^k$, there exists a $\underline{c}$-feasible potential-vector $\underline{\pi} : V \to \mathbf{R}^k$ if and only if $\underline{c}$ is conservative. If $\underline{c}$ is integer vector-valued and conservative, then a $\underline{c}$-feasible $\underline{\pi}$ can be chosen to be integer vector-valued. ■

For $z \in \ddot{\ddot{Q}}$, let $D_z = (V, A_z)$ denote the standard auxiliary digraph associated with $z$, that is,

$$A_z := \{uv : uv \in A,\ z(uv) < g(uv)\} \\ \cup \{vu : uv \in A,\ z(uv) > f(uv)\}.$$

An edge $uv \in A_z$ is called a forward edge when $z(uv) < g(uv)$ and a backward edge when $z(vu) > f(vu)$. We call a di-circuit $C$ of $D_z$ $z$*-improving* if $z' \in \ddot{\ddot{Q}}$ is decreasingly smaller than $z$ on $F$, where $z'(uv)$ is defined for $uv \in A$ as follows:

$$z'(uv) := \begin{cases} z(uv) + 1 & (uv\text{: a forward edge of } C), \\ z(uv) - 1 & (vu\text{: a backward edge of } C), \\ z(uv) & (\text{otherwise}). \end{cases}$$

Let $F_z$ denote the subset of $A_z$ corresponding to $F$ and define a function $z^*$ on $F_z$ as

$$z^*(uv) := \begin{cases} z(uv) & \text{if } uv \in F_{\mathbf{f}}, \\ z(vu) - 1 & \text{if } uv \in F_{\mathbf{b}}, \end{cases}$$

where $F_{\mathbf{f}}$ and $F_{\mathbf{b}}$ denote the sets of forward and backward edges in $F_z$, respectively. Let $\gamma_1 > \gamma_2 > \cdots > \gamma_k$ denote the distinct values of the components of $z^*$. Let $\underline{\varepsilon}_i$ denote the $k$-dimensional $i$-th unit-vector $(0, \dots, 0, 1, 0, \dots, 0)$. We assign a $k$-dimensional vector $\underline{c}(e)$ to every edge $e$ of $D_z$ as follows:

$$\underline{c}(e) := \begin{cases} \underline{0} & \text{if } e \in A_z - F_z, \\ \underline{\varepsilon}_i & \text{if } e \in F_{\mathbf{f}} \text{ and } z^*(e) = \gamma_i, \\ -\underline{\varepsilon}_i & \text{if } e \in F_{\mathbf{b}} \text{ and } z^*(e) = \gamma_i. \end{cases}$$

**Theorem 3.** For an element $z \in \ddot{\ddot{Q}} = \ddot{\ddot{Q}}(f, g; m)$, the following properties are equivalent.
(A) $z$ is decreasingly minimal on $F$.
(B) There is no $z$-improving di-circuit in $D_z$.
(C) There is an integer-valued potential-vector $\underline{\pi}$ such that $\underline{\pi}(v) - \underline{\pi}(u) \preceq \underline{c}(uv)$ for all $uv \in A_z$. ■

## 5. Description of dec-min flows

The next theorem gives a description of all dec-min flows in terms of an appropriate pair $(f^*, g^*)$ of lower and upper bound functions on $A$. The pair has a remarkable property that $0 \leq g^*(e) - f^*(e) \leq 1$ for every edge $e \in F$.

**Theorem 4.** Assume that an $F$-dec-min flow exists. There exists a pair $(f^*, g^*)$ of integer-valued functions on $A$ such that an integral $(f, g)$-bounded $m$-flow $z$ is decreasingly minimal on $F$ if and only if $z$ is an integral $(f^*, g^*)$-bounded $m$-flow. Here the pair $(f^*, g^*)$ satisfies: (i) $f \leq f^* \leq g^* \leq g$, (ii) $f^*(e), g^*(e) \in \mathbf{Z}$ and $0 \leq g^*(e) - f^*(e) \leq 1$ for $e \in F$, and (iii) $f^*(e) \in \mathbf{Z} \cup \{-\infty\}$ and $g^*(e) \in \mathbf{Z} \cup \{+\infty\}$ for $e \in A - F$. ■

Such a pair $(f^*, g^*)$ can be computed by a strongly polynomial algorithm, which is built on the discrete Newton–Dinkelbach algorithm and standard network flow algorithm.

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