日本オペレーションズ・リサーチ学会
2019年 秋季研究発表会

A new random projection method for Linear Conic Programming

	RIKEN-AIP		*POIRION Pierre-Louis
05000745	東京大学	ロウレンソ ブルノ フィゲラ	LOURENÇO Bruno Figueira
01308490	東京大学/RIKEN-AIP	武田朗子	TAKEDA Akiko

1. Introduction

In this paper we propose a new random projection method that allows to reduce the size of a Conic Linear Program (CLP). More precisely, we consider the case when the underlying cone is homogeneous; we project the constraints of the problem into a lower dimensional face of the cone while preserving, approximately, the optimal value of the LP. This extends the work in [4], where the authors considered an LP with equality constraints and proved that we can build a new LP, whose optimal value approximates the original one, with much fewer constraints by randomly aggregating them using a matrix of independent and identically distributed (iid) random variables. The LP case with inequality constraints has also been considered previously. The main tool of the above paper is the Johnson-Lindenstrauss Lemma [1], which states that a set of high-dimensional points can be projected to a much lower dimensional one, while preserving, approximately, the Euclidean distance between these points. In this talk we use a characterization of homogeneous cones, given by Vindberg [3], to extend random projections to homogeneous cones.

Let us consider the following conic LP:

$$\left.\begin{array}{lc} \min\limits_{x\in\mathbb{R}^n} & c^\top x \\ & A\,x-b\in\mathcal{C} \end{array}\right\} P$$

where $c\in\mathbb{R}^n, b\in\mathbb{R}^m$, $A\in\mathbb{R}^{m\times n}$ and $\mathcal{C}$ is an homogeneous cone contained in $\mathbb{R}^m$, that is a full dimensional proper convex cone such that the group of linear automorphisms of the cone acts transitively on its interior. It has been proven in [3] that such a cone is isomorphic to the cone $K(\mathcal{A})$ generated by an T-algebra $\mathcal{A}$.

A T-algebra of rank r is an matrix algebra $\mathcal{A}=\bigoplus_{i,j=1}^r\mathcal{A}_{ij}$, that is an algebra satisfying:

$$\mathcal{A}_{ij}\mathcal{A}_{lk}=\begin{cases}\mathcal{A}_{ik} & \text{if } j=l\\ \{0\} & \text{otherwise}\end{cases}$$

where $\mathcal{A}_{ij}$ is a real vector space of dimension n_{ij}. A matrix algebra has an involution *, that is, an endomorphism of $\mathcal{A}$ such that:

- $a^{**}=a$.
- $(ab)^*=b^*a^*$.
- $\mathcal{A}_{ij}^*\subseteq\mathcal{A}_{ji}$.

A T-algebra must further satisfy the following axioms:

- $\mathcal{A}_{ii}$ is isomorphic to $\mathbb{R}$ through an isomorphism ρ_i. We denote by e_i its unit element.
- $e_ia_{ij}=a_{ij}e_j=a_{ij}$ for all $a_{ij}\in\mathcal{A}_{ij}$.
- $tr(ab)=tr(ba)$ where $tr(\cdot)$ is the trace operator defined by $tr(a)=\sum\limits_{i=1}^r n_i\rho_i(a_{ii})$ with $n_i=1+\frac{1}{2}\sum\limits_{j\neq i}n_{ij}$.
- $tr(a(bc))=tr((ab)c)$.
- $tr(aa^*)>0$ unless $a=0$.
- $t(uv)=(tu)v$ for all $t,u,v\in\mathcal{T}$ where $\mathcal{T}$ is the set of upper-triangular matrices.
- $t(uu^*)=(tu)u^*$ for all $t,u\in\mathcal{T}$.

Let $\mathcal{T}^+$ denotes the set of upper-triangular matrices whose diagonal elements are positive. The cone, $K(\mathcal{A})$ generated by $\mathcal{A}$ is defined by

$$K(\mathcal{A})=\{tt^*\mid t\in\mathcal{T}^+\}.$$

Using the T-algebra characterization of homogeneous cone, we assume that m is the dimension of the algebra $\mathcal{A}$ and that $\mathcal{C}=K(\mathcal{A})$.

We define the projected problem P_S as the following linear conic problem:

$$\left.\begin{array}{lc} \min\limits_{x\in\mathbb{R}^n} & c^\top x \\ & S(A\,x-b)S^*\in\mathcal{C} \end{array}\right\} P_S$$

where the $m \times m$ matrix $S = \begin{pmatrix} 0 \\ U \end{pmatrix}$ is a matrix of $\mathcal{T}$ such that the $k \times m$ matrix U is random.

The goal of this talk is to prove that we can build some random matrix S such that we can approximate the value of P by the value of P_S.

2. Random projections

In this section we recall some definitions and facts about concentration inequalities.

定義 1. *We say that a zero mean random variable X is sub-gaussian if there exists $K > 0$ such that for all $t > 0$:*

$$P(|X| > t) \leq 2e^{-\frac{t^2}{K^2}} \tag{1}$$

The sub-gaussian norm, $\|X\|_{\psi_2}$, of X is defined to be the smallest K satisfying (1).

定義 2. *We say that a zero mean random variable X is sub-exponential if there exists $K > 0$ such that for all $t > 0$:*

$$P(|X| > t) \leq 2e^{-\frac{t}{K}} \tag{2}$$

The sub-exponential norm, $\|X\|_{\psi_1}$, of X is defined to be the smallest K satisfying (2).

Sub-gaussian and sub-exponential random variables are closely related, as we can see any sub-gaussian random variable is also sub-exponential, furthermore, it turns out that the product of two sub-gaussian random variables is sub-exponential.

We now recall the famous Johnson-Lindenstrauss lemma.

Lemma 2.1 (Johnson-Lindentrauss Lemma [2])**.** *Let $\mathcal{X}$ be a set of n points in $\mathbb{R}^m$ and let G be a $k \times m$ random matrix whose rows are independent, mean zero, sub-gaussian and isotropic, then with probability $1 - 2\exp(-\mathcal{C}_0 k\varepsilon^2)$, we have that for all $x_i, x_j \in \mathcal{X}$*

$$(1-\varepsilon)\|x_i - x_j\|_2 \leq \frac{1}{\sqrt{k}}\|Gx_i - Gx_j\|_2 \leq (1+\varepsilon)\|x_i - x_j\|_2 \tag{3}$$

where $\mathcal{C}_0$ is an universal constant.

Notice that this lemma implies that

$$k = O\left(\frac{\log(n)}{\varepsilon^2}\right)$$

3. Random projection matrices S over $K(\mathcal{A})$

In this section, we first extend the results of the previous section: we consider a random matrix of $\mathcal{T}$

$$S = \begin{pmatrix} 0 \\ U \end{pmatrix},$$

where U_{ij} is a random element of $\mathcal{A}_{ij}$ drawn from the normal distribution $\mathcal{N}(0, \Sigma_{ij})$ where Σ_{ij} is a positive definite matrix over the euclidean space $\mathcal{A}_{ij}$ which depends on the bilinear form $\mathcal{A}_{ij} \times \mathcal{A}_{ji} \mapsto \mathcal{A}_{ii}$ and where U_{ii} is the absolute value of a random variable built from the normal distribution on $\mathbb{R}$.

Then we obtain some asymptotic approximation results on the value of the projected problem P_S. More precisely, we prove that when m is large enough we have

$$v(P) - \varepsilon f(P) \leq v(P_S) \leq v(P),$$

where $v(\cdot)$ denotes the value of an optimization problem, ε is a parameter that control the size of the projected problem P_S, and where $f(P)$ is a function that depends on some parameters of the problem P.

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