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Random Assignment: Characterizing the Extended Serial Rule

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1. Introduction

We consider the problem of allocating a set of indivisible goods to agents when monetary transfers are not allowed and agents reveal only ordinal preferences. Recently, we generalized the *probabilistic serial* mechanism of a seminal paper [1] given by Bogomolnaia and Moulin, where we allow the *interchangeability among the goods*, provided they subject to submodular constraints. Based on the results given by Hashimoto et al. [4], we show that the *extended probabilistic serial* rule given by Fujishige et al. in [3] is the only mechanism satisfying *ordinal fairness* and *non-wastefulness*, where *non-wastefulness* is newly defined and simple.

2. Model

Let $N = \{1, 2, \cdots, n\}$ be a set of *agents* and E be a set of *goods*. For each $i \in N$ we assume a linear order *preference* $\succ_i$ on E. Let $\mathcal{L}$ be all the profile of preferences $\succ (\succ_i, i \in N)$.

A set function $\rho : 2^E \to \mathbb{R}_{\geq 0}$ with $\rho(\emptyset) = 0$ is called a *submodular function* if for any $X, Y \in 2^E$ we have $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$. We also suppose the monotonic condition, i.e., for any $X, Y \in 2^E$ with $X \subseteq Y$ we have $\rho(X) \leq \rho(Y)$. For (E, ρ), the *base polyhedron* is the polytope satisfying

$$\mathrm{B}(\rho) = \{x \in \mathbb{R}^E \mid \forall X \subset E : x(X) \leq \rho(X), \\ x(E) = \rho(E)\}.$$

We have $\mathrm{B}(\rho) \neq \emptyset$, see more details about submodular functions in [2].

Now we denote the random assignment problem by $\mathbf{RA} = (N, E, \mathcal{L}, \mathbf{d}, (E, \rho))$, where $\mathbf{d} = (d(i) \mid i \in N) \in \mathbb{Z}^N_{>0}$ is agents' total *demand vector* (see Figure 1).

Given $\mathbf{RA} = (N, E, \mathcal{L}, \mathbf{d}, (E, \rho))$, an *random assignment*, also called an *expected allocation*, is a nonnegative matrix $P \in \mathbb{R}^{N \times E}_{\geq 0}$ satisfying

(i) $\sum_{e \in E} P(i, e) \leq d(i)$ for all $i \in N$,
(ii) $\sum_{i \in N} P^i \in \mathrm{B}(\rho)$,

where P^i is the ith row vector of P, representing goods allocated to the agent i. We assume $\sum_{i \in N} d(i) \geq \sum_{i \in N} \sum_{e \in E} P(i, e) = \rho(E)$.

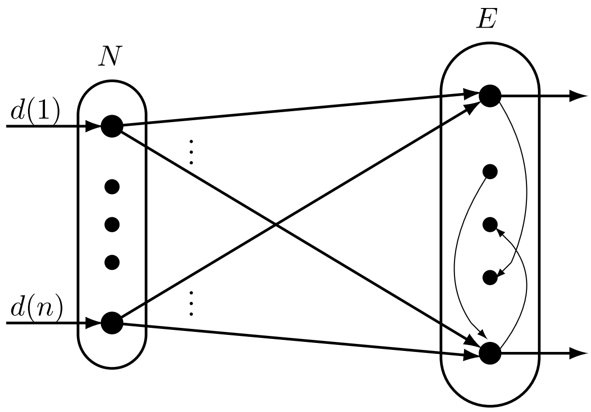

Figure 1: A polymatroid constraint of network type (the interchangeability among goods).

3. Extended Probabilistic Serial Mechanism (EPS)

Extended probabilistic serial mechanism:[1] This is an extension of the mechanism introduced by Bogomolnaia and Moulin [1], allocating goods via the following *simultaneous eating algorithm* (SEA).

Given a problem $\mathbf{RA}$, think of each good as an infinitely divisible object subjected to submodular constraints that agents eat in some time interval.

Step 1. Each agent eats away from her favorite goods at the same unit speed. Proceed to the next step when some goods are bounded, or saturated, by submodular constraints.

Step $s > 1$. Each agent eats away from her remaining favorite goods at the same unit speed. Proceed to the next step when some goods are saturated by submodular constraints.

4. Ordinally Fair Non-Wasteful Characterization

Given $\mathbf{RA} = (N, E, \mathcal{L}, \mathbf{d}, (E, \rho))$:

Definition 1: An expected allocation P is *non-wasteful* at $\succ$ if for all $i \in N$ and all $e \in E$ such that $P(i, e) > 0$, we have $\sum_{e' \succ_i e} \sum_{j \in N} P(j, e') = \rho(\{e' \mid e' \succ_i e\})$, i.e., being saturated by submodular constraints.

Let $U(\succ_i, e) := \{e' \in E \mid e' \succeq_i e\}$ be the *upper*

[1] The original description is from paper [4].

contour set of goods e at $\succ_i$. For an expected allocation P, let $F(\succ_i, e, P^i) = \sum_{e' \in U(\succ_i, e)} P(i, e')$.

Definition 2: For a preference file $\succ \in \mathcal{L}$, an expected allocation P is *ordinally fair* at $\succ$ if for all $e \in E$ and all $i, j \in N$ with $P(i, e) > 0$, we have $F(\succ_i, e, P^i) \leq F(\succ_j, e, P^j)$.

Now we have the theorem of characterizing EPS, whose proof is a direct adaptation of the one for random assignment problem given by Hashimoto et al. [4, Theorem 1].

Theorem 4.1: *A mechanism is ordinally fair and non-wasteful if and only if it is* EPS.

In the proof of Theorem 4.1, the main difference from the one of paper [4] is the non-wastefulness.

Proposition 4.2: EPS *is ordinally fair and non-wasteful.*

(Proof) By similar discussions as the corresponding proof of Theorem 1 in paper [4]. □

Let P be an expected allocation. Define $\pi(e) = \min_i F(\succ_i, e, P^i)$ for all $e \in E$. Group goods with the same value of π. Label the partition of E as $T_1, \cdots, T_p$ so that $\pi(e \mid e \in T_s) < \pi(e \mid e \in T_{s+1})$ for all $s \leq p-1$. Let $S_0 = \emptyset$ and $S_s = T_1 \cup \cdots \cup T_s$. For $\bar{e} \in E \setminus S_{s-1}$, define $N_s(\bar{e}) = \{i \in N \mid \bar{e} \succeq_i e', \forall e' \in E \setminus S_{s-1}\}$, and denote $\bar{e}_s$ if we restrict $\bar{e}_s$ further on T_s.

Lemma 4.3: *If P is an ordinally fair expected allocation, then for all $s \leq p$, and $k \notin N_s(\bar{e}_s)$ we have $P(k, \bar{e}_s) = 0$, and $i \in N_t(\bar{e}_s)$ we have $F(\succ_i, \bar{e}_s, P^i) = \pi(\bar{e}_s)$.*

(Proof) By similar discussions as the proof, Steps 2 and 3, of Theorem 1 of paper [4]. □

Lemma 4.4: *If P is an ordinally fair expected allocation and non-wasteful, then we have $\sum_{\bar{e}_s \in T_s} \sum_{j \in N_s(\bar{e}_s)} P(j, \bar{e}_s) = \rho(S_s) - \rho(S_{s-1})$ for all $s < p$.*

(Proof) We prove it by induction. The assumption holds vacuously true for $S_0 = \emptyset$. Suppose that the claim is true for $s-1$, we show it for s.

For each $i \in N$, let $\hat{e}$ be the most preferred good in $E \setminus S_s$ with $P(i, \hat{e}) > 0$. Such $\hat{e}$ exists since $s < p$. Suppose $i \in N_s(\bar{e}_s)$, since $\hat{e}$ is agent i's most preferred goods in $E \setminus S_s$ with $P(i, \hat{e}) > 0$, we have that $F(\succ_i, b, P^i) < \pi(\bar{e}_s)$ if $b \succ_i \bar{e}_s$ and $F(\succ_i, b, P^i) = \pi(\bar{e}_s)$ if $\bar{e}_s \succ_i b \succ_i \hat{e}$. Hence, we have $\{e' \mid e' \succ_i \hat{e}\} \subseteq S_s$, and

$$\bigcup_{\bar{e}_s \in T_s} \{e' \mid i \in N_s(\bar{e}_s), e' \succ_i \hat{e}, \} \bigcup S_{s-1} = S_s.$$

Summarizing above discussions, we have

$$\begin{aligned} \sum_{\bar{e}_s \in T_s} \sum_{j \in N_s(\bar{e}_s)} P(j, \bar{e}_s) &= \sum_{\bar{e}_s \in T_s} \sum_{j \in N} P(j, \bar{e}_s) \\ &= \rho(S_s) - \rho(S_{s-1}), \end{aligned}$$

where the first equality from Lemma 4.3, the second one from above equality relation just proved, inductive assumption, and the property of submodular functions, the union of tight sets is again tight set. □

Remark 1: Note that by combining the ordinal fairness and the property of submodular functions, our definition of non-wastefulness is also simple.

(Proof of Theorem 4.1) We argue by induction.

We assume that for all $t < s$, **(i)** for all $i \in N_t(\bar{e}_t)$, we have $F(\succ_i, \bar{e}_t, P^i) = F(\succ_i, \bar{e}_t, \mathrm{EPS}^i) = \pi(\bar{e}_t)$ **(ii)** for all $k \notin N_t(\bar{e}_t)$, we have $P(k, \bar{e}_t) = \mathrm{EPS}(k, \bar{e}_t) = 0$.

By similar arguments as the proof, Steps 3 and 4, of Theorem 1 [4], we can obtain

$$\begin{aligned} &\sum_{p \leq s} \sum_{\bar{e}_p \in T_p} \sum_{j \in N_p(\bar{e}_p)} P(j, \bar{e}_p) \\ &< \rho(T_1 \cup \cdots \cup T_s) = \rho(S_s). \end{aligned}$$

The above inequality contradicts to $\sum_{i \in N} P^i \in \mathrm{B}(\rho)$ if $s = p$, otherwise it violates the the claim of Lemma 4.4. Therefore, P coincides EPS. □

Remark 2: The property of the base $\mathrm{B}(\rho)$ of a polymatroid (E, ρ) is crucial. The base can be reached from every local *greedy* direction, or local *ordinally fair* direction.

References

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