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Optimal execution strategies with generalized price impacts in a continuous–time setting

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1. Introduction

In this article, we analyze a continuous–time analog of the optimal trade execution problem with generalized price impacts, which was recently discussed in Ohnishi and Shimoshimizu (2019) for a discrete–time setting. Our problem is formulated as a stochastic continuous control problem over a finite horizon of maximizing the expected utility from the final wealth of the large trader with Constant Absolute Risk Aversion (CARA) von Neumann–Morgenstern (vN-M) utility function. By examining the Hamilton–Jacobi–Bellman (HJB) equation, we characterize the optimal value function and optimal trade execution strategy, and conclude that the trade execution strategy is a time–dependent affine function of three state variables: the remained trade execution volume of the large trader, the residual effects of past price impacts caused by both of the large trader and other small traders, and the aggregate volume of instantaneous trade orders submitted by small traders.

2. Market Model

We assume a large trader with CARA utility function where the risk–averse rate $\gamma > 0$ in a market. He/she purchases $\mathfrak{Q}$ $(\in \mathbb{R})$ volume of a risky asset in the time window $[0, T]$. Let Q_t $(\in \mathbb{R})$ be the cummulative purchase up to time $t \in [0, T]$ of a large trader. Then, the number of shares remained to purchase at time t is described as $\overline{Q}_t = \mathfrak{Q} - Q_t$, with the initial and terminal conditions: $\overline{Q}_0 = \mathfrak{Q}$ and $\overline{Q}_T = 0$.

We consider a continuous trading strategy:

$$\mathrm{d}Q_t = \dot{Q}_t \mathrm{d}t. \tag{1}$$

Here it is assumed that Q_t is continuously differentiable in time t. We denote by the positive and negative $\dot{Q}_t$ the acquisition and liquidation of the risky asset, respectively. This leads to a similar setup for a selling problem.

The execution price of an asset $\widehat{P}$ is assumed to follow a linear price impact model as follows:

$$\widehat{P}_t = P_t + \lambda_t \dot{Q}_t + \kappa_t v_t, \tag{2}$$

where P_t represents the market price of the asset at time t, and λ_t is a price impact coefficient at time t. Here v_t represents the aggrergate volume of instantaneous order submitted by small traders (or noise traders) and κ_t the price impact per unit at time t caused by the submission of small traders. Note that both of λ_t and κ_t are assumed to be deterministic functions of time t.

The large trader's wealth process at time t, denoted by W_t, evolves as

$$\mathrm{d}W_t = -\widehat{P}_t \dot{Q}_t \mathrm{d}t = -\left(P_t + \lambda_t \dot{Q}_t + \kappa_t v_t\right) \dot{Q}_t \mathrm{d}t. \tag{3}$$

Besides the above temporary price impact, we assume a permanent and a transient parts of market impact.

The residual effect of the transient part is defined, with the deterministic linear temporary impact coefficient α_t, as

$$\mathrm{d}R_t = -\rho R_t \mathrm{d}t + \alpha_t \left(\lambda_t \dot{Q}_t + \kappa_t v_t\right) \mathrm{d}t. \tag{4}$$

where $\rho > 0$ stands for the deterministic resilience speed and we asume $R_0 = 0$. Note that $\alpha_t(\lambda_t \dot{Q}_t + \kappa_t v_t)$ represents the temporary price impact.

The market price is assumed to consist of the sum of two components:

$$P_t = P_t^f + R_t, \tag{5}$$

where P_t^f stands for the fundamental price defined by

$$\mathrm{d}P_t^f = \beta_t \left(\lambda_t \dot{Q}_t + \kappa_t v_t\right) \mathrm{d}t + \mathrm{d}Z_t. \tag{6}$$

Note that $\beta_t(\lambda_t \dot{Q}_t + \kappa_t v_t)$ represents the permanent price impact. The other term Z_t represents the effect of some public news/information about the economic situation which may affect the market price (or quoted price) defined as follows:

$$\mathrm{d}v_t = \left(a_t^v - b_t^v v_t\right) \mathrm{d}t + \sigma_t^v \mathrm{d}B_t^v; \tag{7}$$

$$\mathrm{d}Z_t = \mu_t^Z \mathrm{d}t + \sigma_t^Z \mathrm{d}B_t^Z, \tag{8}$$

where B_t^v and B_t^Z stand for two standard Brownian motions with $B_0^v = 0$, $B_0^Z = 0$, and we assume that the quadratic co–variation of B_t^v and B_t^Z is given:

$$\mathrm{d}\langle B^v, B^Z \rangle_t = \rho^{v,Z} \mathrm{d}t, \tag{9}$$

which implies that these two processes are correlated with each other. Note that $a_t^v, b_t^v, \sigma_t^v, \mu_t^Z, \sigma_t^Z$ in the above dynamics of v_t and Z_t are all deterministic in time t.

Finally, the set $\mathcal{A}$ of admissible execution strategies consists of cumulative executed processes $\{Q_t\}_{t\in[0,T]}$ which are adapted to the filtration generated by $\{(v_t, Z_t)\}_{t\in[0,T]}$ and have continuously differentiable paths with $Q_0 = 0$, $Q_T = \mathfrak{Q}$.

3. Optimal Value Function and HJB Equation

In this section, we derive a HJB equation of the problem, from which we characterize an optimal execution strategy and the optimal value function. For that purpose, we first define the state of the controlled process at time $t \in [0, T]$. The controlled state, denoted by s_t, is a 5–tuple and is defined as

$$s_t := \big(W_t, P_t, \overline{Q}_t, R_t, v_t\big) \in \mathbb{R}^5 =: S. \tag{10}$$

The utility payoff (or reward) arises only from the terminal wealth at the maturity:

$$g_T(s_T) := \begin{cases} -\exp\{-\gamma W_T\} & \text{if } \overline{Q}_T = 0; \\ -\infty & \text{if } \overline{Q}_T \neq 0, \end{cases} \tag{11}$$

from which we define the (conditional) expected utility of the large trader at time $t \in [0, T]$ on an execution strategy $Q = \{Q_t\}_{0\le t\le T}$ as

$$V_t^Q := \mathbb{E}\left[g_T(s_T^Q)\Big|\mathcal{F}_t\right], \tag{12}$$

where the superscript Q of s_T^Q emphasizes the dependence on Q explicitly.

Let the optimal (expected utility) value from time $t \in [0, T]$ by

$$V_t := \operatorname*{ess\,sup}_{Q\in\mathcal{A}} V_t^Q. \tag{13}$$

Then, V_t depends on the past history or information only through the (controlled) state: $s_t = \big(W_t, P_t, \overline{Q}_t, R_t, v_t\big) \in S = \mathbb{R}^5$ and we denote this functional dependence by the optimal value function as

$$V\left[t, W_t, P_t, \overline{Q}_t, R_t, v_t\right] := V_t. \tag{14}$$

If we assume that the function $V\colon [0, T] \times S \longrightarrow \overline{\mathbb{R}}$ is in $\mathbb{C}^{1,2}$, that is, V is continuously differentiable with respect to time t and continuously twice differentiable with respect to each state variable, it satisfies, from the dynamic programming principle, the following HJB equation: for $0 \le t < T$,

$$\begin{aligned}
&\sup_{\dot{Q}_t\in\mathbb{R}} [-(P_t + \lambda_t\dot{Q}_t + \kappa_t v_t)\dot{Q}_t\partial_W V \qquad (15)\\
&+ (\alpha_t + \beta_t)\lambda_t\dot{Q}_t\partial_P V - \dot{Q}\partial_{\overline{Q}} V + \alpha_t\lambda_t\dot{Q}_t\partial_R V]\\
&+ \partial_t V + \{-\rho R_t + (\alpha_t + \beta_t)\kappa_t\mu_t^v + \mu_t^Z\}\partial_P V\\
&+ (-\rho R_t + \alpha_t\kappa_t\mu_t^v)\partial_R V + (a_t^v - b_t^v v_t)\partial_v V\\
&+ \frac{1}{2}\big\{(\sigma_t^Z)^2\partial_{PP}V + 2\sigma_t^v\sigma_t^Z\rho^{v,Z}\partial_{Pv}V + (\sigma_t^v)^2\partial_{vv}V\big\} = 0
\end{aligned}$$

with the terminal condition:

$$V[T, W_T, P_T, \overline{Q}_T, R_T, v_T] = g_T(T, W_T, P_T, \overline{Q}_T, R_T, v_T). \tag{16}$$

Then, we can characterize the optimal execution strategy and its associated value function of Eq. (14) explicitly by appropriately guessing the form of the optimal value function and verifying the obtained solution. The optimal dynamic execution strategy is derived by solving the above HJB equation (16).

We obtain the following main theorem.

Theorem 3.1 Under a set of regularity conditions:

(1) The optimal execution volume at time $t \in [0, T]$, denoted as $\dot{Q}_t^*$, becomes an affine function of the remaining execution volume $\overline{Q}_t$, the cumulative residual effect R_t at time t, and the aggregate volume v_t of instantaneous trade orders submitted by small traders:

$$\dot{Q}_t^* = f_t(s_t) = a_t + b_t\overline{Q}_t + c_t R_t + d_t v_t. \tag{17}$$

(2) The optimal value function $V[t, W_t, P_t, \overline{Q_t}, R_t, v_t]$ at time $t \in [0, T]$ is represented as follows:

$$\begin{aligned}
&V[t, W_t, P_t, \overline{Q}_t, R_t, v_t] \qquad (18)\\
&= -\exp\{-\gamma\big[W_t - P_t\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + I_t\overline{Q}_t R_t\\
&+ J_t R_t^2 + L_t R_t + M_t\overline{Q}_t v_t + N_t R_t v_t + X_t v_t^2 + Y_t v_t + Z_t\big]\},
\end{aligned}$$

where $a_t, b_t, c_t, d_t;\ G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, Z_t$ for $t \in [0, T]$ are deterministic functions of time t which are dependent on the model parameters, and these are assumed to exist as a unique solution of a simultaneous system of ordinary differential equations.

Corollary 3.1 If the orders v_t posed by small traders for $t \in [0, T]$ are deterministic, the optimal execution volumes $\dot{Q}_t^*$ at time $t \in [0, T]$ also become a deterministic function of time t, and thus it is in a class of the static (and non-randomized) execution strategies.

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References

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